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DESIGN DEVELOPMENT AND DURABILITY VALIDATION OF POSTBUCKLED COMPOSITE AND METAL PANELS

VOLUME III - ANALYSIS AND TEST RESULTS

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The objective of this program was to develop design procedures and durability validation methods for curved metal and composite panels designed to operate in the postbuckling range under the action of combined compression and shear loads. This research and technology effort was motivated by the need to develop design and life prediction methodologies for such structures.

The program has been organized in four tasks. In Task I, Technology Assessment, a complete review of the available test data was conducted to establish the strength, durability, and damage tolerance characteristics of postbuckled metal and composite panels and to identify data gaps that need to be filled. Task II, Data Base Development, was comprised of static and fatigue tests required to fill in the data gaps identified in Task I. New rigorous static analysis methods aimed at improving the accuracy of the existing semi-empirical analyses and life prediction techniques were developed in Task III. Task IV consisted

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of technology consolidation where the results of this program were incorporated in the Preliminary Design Guide developed under Contract F33615-81-C-3208 to provide a comprehensive design guide for postbuckled aircraft structures. The comprehensive design guide was also exercised in this task, on an actual aircraft fuselage section to illustrate the methodology and demonstrate weight and cost trade-offs.

This final report consists of the following five volumes:

Volume I - Executive Summary

Volume II - Test Results

Volume III - Analysis and Test Results

Volume IV - Design Guide Update

Volume V - Automated Data Systems Documentation

PREFACE

The work documented in this report was performed by Northrop Corporation, Aircraft Division, Hawthorne, California, under Contract F33615-84-C-3220 sponsored by the Air Force Wright Aeronautical Laboratories, Flight Dynamics Laboratory, WRDC/FIBE. The work was performed in the period from September 1984 through April 1989. The Air Force Program Monitor was Dr. G. P. Sendeckyj.

The following Northrop personnel contributed to the performance of the contract in their respective areas of responsibility:

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Dr. N. M. Bhatia	Testing and Data Correlation
M. Kerbow	Testing
R. Cordero	Data Analysis/Graphics
R. Urias	Documentation

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INTRODUCTION

1.1 BACKGROUND

Several recent studies have demonstrated that the structural efficiency of military and commercial aircraft can be improved by taking advantage of the postbuckled strength of stiffened panels. An assessment of the current postbuckled stiffened panel design, analysis and applications technology (References 1 and 2) shows that several deficiencies need to be addressed to establish a systematic postbuckling design methodology. In References 1 and 3 a design and analysis methodology was developed for flat and curved stiffened panels made of either composite or metallic materials, and subjected to either compression loading or shear loading. In practice, however, stiffened airframe panels are subjected to a combination of axial compression and shear loads. A semi-empirical design methodology for curved metal panels under combined loading exists (Reference 4) but has seen limited verification. The present program was undertaken to extend the Reference 1 and Reference 4 methods for application to curved composite panels under combined uniaxial compression and shear loading, and to further substantiate the metal panel design procedures.

1.2 PROGRAM OBJECTIVES

The overall objectives of the program were to develop validated design procedures and an analysis capability for curved metal and composite postbuckled panels under combined uniaxial compression and shear loading. The specific requirements encompassed by these objectives were as follows:

1. Extend the existing semi-empirical analysis methodology (Reference 1) into a design tool for curved composite and metal panels subjected to combined uniaxial compression and shear loading. Account for any unique failure modes.

2. Develop a more rigorous energy method based analysis to predict the displacement and stress fields in postbuckled panels.

3. Develop a static and fatigue data base for composite and metal panel design verification.

4. Develop a fatigue analysis method for metal panels.

5. Prepare a procedural design guide. Exercise the design guide on a realistic aircraft component.

The work performed to accomplish these objectives is documented in this report.

1.3 PROGRAM SUMMARY

The program approach and plan paralleled those in Reference 1. At the onset, a technology review was conducted to update the data base and to clearly define the deficiencies in the static strength, durability and damage tolerance design and analysis of postbuckled metal and composite panels. The durability and damage tolerance technology assessment is documented in Reference 2. As a result of this technology assessment, a semi-empirical design methodology for curved panels under combined loading was established and a verification test program was planned. In addition, an energy method based approach to predict the static response of postbuckled stiffened panels was formulated. The technology assessment was accomplished in Task I of the program. A summary of the data gaps identified by this technology assessment is summarized in Table 1. In Task II, Data Base Development, the tests required to fill the data gaps identified in Task I were conducted. Analytical model development and verification was accomplished in Task III. Task IV consisted of technology consolidation where the results of the program were incorporated in a Design Guide (Reference 5) for postbuckled structures.

The composite and metal panels tested in the program were cylindrically curved and identical to the shear panels tested in Reference 1. The design methodology initially established from the technology assessment was used to estimate the buckling and postbuckling load capacities of the panels.

Table 1. Summary of Technology Gaps Identified in Task I.

TECHNOLOGY	COMPOSITE PANELS	METAL PANELS
DESIGN FOR STATIC STRENGTH	• Semiempirical design methodology for flat and curved panels under combined loads • Stiffener skin separation prediction under combined loads • Buckling load interaction prediction • Identification of failure modes under combined loading	• Semiempirical design methodology verification • Skin permanent-set failure criterion
DURABILITY	• Fatigue test data for curved composite panels under combined loads. Influence of R-Ratio; influence of shear to compression load ratio. Identify fatigue failure modes under combined loading and obtain S-N data. • Analysis Methodology to predict strain energy release rate for disbond growth at the stiffener/web interface • Simple tests to measure critical strain energy release rate and disbond growth rate as a function of the strain energy release rate	• Fatigue test data for curved metal panels to identify fatigue failure modes and generate S-N curves • Influence of combined load ratio on fatigue behavior • Life prediction methodology for curved and flat panels under combined loading
DAMAGE TOLERANCE	• Influence of impact damage under combined loads - Mid-bay location - Over stiffener location	• Analyses for compliance with MIL-A-83444

Static and fatigue tests including several static strain surveys were conducted on the test articles. The static test data were used to verify the semi-empirical design methodology, whereas the fatigue test data were utilized to determine the fatigue failure modes and obtain load versus life data to formulate fatigue analysis approaches. The test results, in conjunction with the semi-empirical design methodology were used to update the Preliminary Design Guide (Reference 3).

1.4 REPORT OUTLINE

This report details the correlations between the results of the semi-empirical and rigorous analyses, and the tests conducted. Section 2 describes the semi-empirical analysis methodology and Section 3 details the development of the energy method based analysis. The actual correlations between the analysis and test results are presented in Section 4.

Volume I of the final report presents an executive summary of the program. The test program details are documented in Volume II of the final report. Correlation between the test data and analyses is presented in Volume III. The Design Guide Update and the Software User's Manual are published separately as Volumes IV and V, respectively.

SECTION 2

SEMI-EMPIRICAL DESIGN METHODOLOGY

2.1 BACKGROUND

An in-depth survey of the semi-empirical design methods for metal panels and their evolution into a design methodology for curved composite panels under shear or uniaxial compression loads is given in Reference 1. The shear panel and compression panel analysis methods of Reference 1 were used as the starting points for the current program. Initially, the interaction rules used for metal panels (References 4 and 6) were adopted to predict buckling under combined shear and compression loading. Test data were then used to verify these rules and suggest modifications where necessary. Postbuckling failure envelopes were developed by accounting for the failure modes possible under shear loading only, and under pure compression loading. Failure predictions under combined loading took into account load interaction for stiffener crippling and skin rupture. This semi-empirical analysis methodology is detailed in the following subsections.

2.2 DESIGN METHODOLOGY

A complete static analysis of postbuckled structures consists of predicting the initial buckling loads, the failure or ultimate load of the panel after buckling, and the local skin and stiffener displacement and stress fields. The latter predictions are especially required for metal panel fatigue analysis. The semi-empirical methodology detailed in this section can be used to obtain the initial buckling and the failure loads. The energy method based analysis described in Section 3 is useful in predicting the local stresses and displacements.

The semi-empirical analysis method was selected as a design tool for postbuckled structures to provide a quick, inexpensive, and reasonably accurate but conservative design methodology. The scope of this program encom-

passed cylindrically curved stiffened panels loaded in uniaxial longitudinal compression and shear. Since the Reference 1 methodology is the basis for the combined loading design procedure, the semi-empirical analysis described in this section applies to cylindrically curved panels under simultaneously acting longitudinal compression and in-plane shear.

The essence of the combined loading design procedure is summarized in Figure 1. As can be seen in the figure, the curved panel is analyzed for compression and shear loads independently according to Reference 1 methods. Buckling loads under combined loading are predicted using the parabolic interaction rule developed for metal panels (References 4 and 6). Failure analysis requires consideration of failure modes under shear and compression acting independently and those due to the interaction of the loads. The failure modes affected by combined loading are stiffener crippling, and skin rupture under tensile loading determined from a principal strain analysis and the maximum strain criterion. The following paragraphs present the detailed equations necessary for a semi-empirical analysis.

2.2.1 Skin Buckling Strain/Load

The shear and compression buckling strains or loads for metal and composite panels are calculated according to the equations given in Reference 1. For continuity these equations are summarized below.

The compression buckling stress for curved metal sheet panels can be calculated from:

$$F_{CR} = \frac{K_c \pi^2 E}{12(1-\nu^2)} \left(\frac{t_w}{b_s}\right)^2 \tag{1}$$

where,

F_{CR} buckling stress, psi

t_w thickness of the skin, in

b_s stiffener spacing measured between fastener lines, in

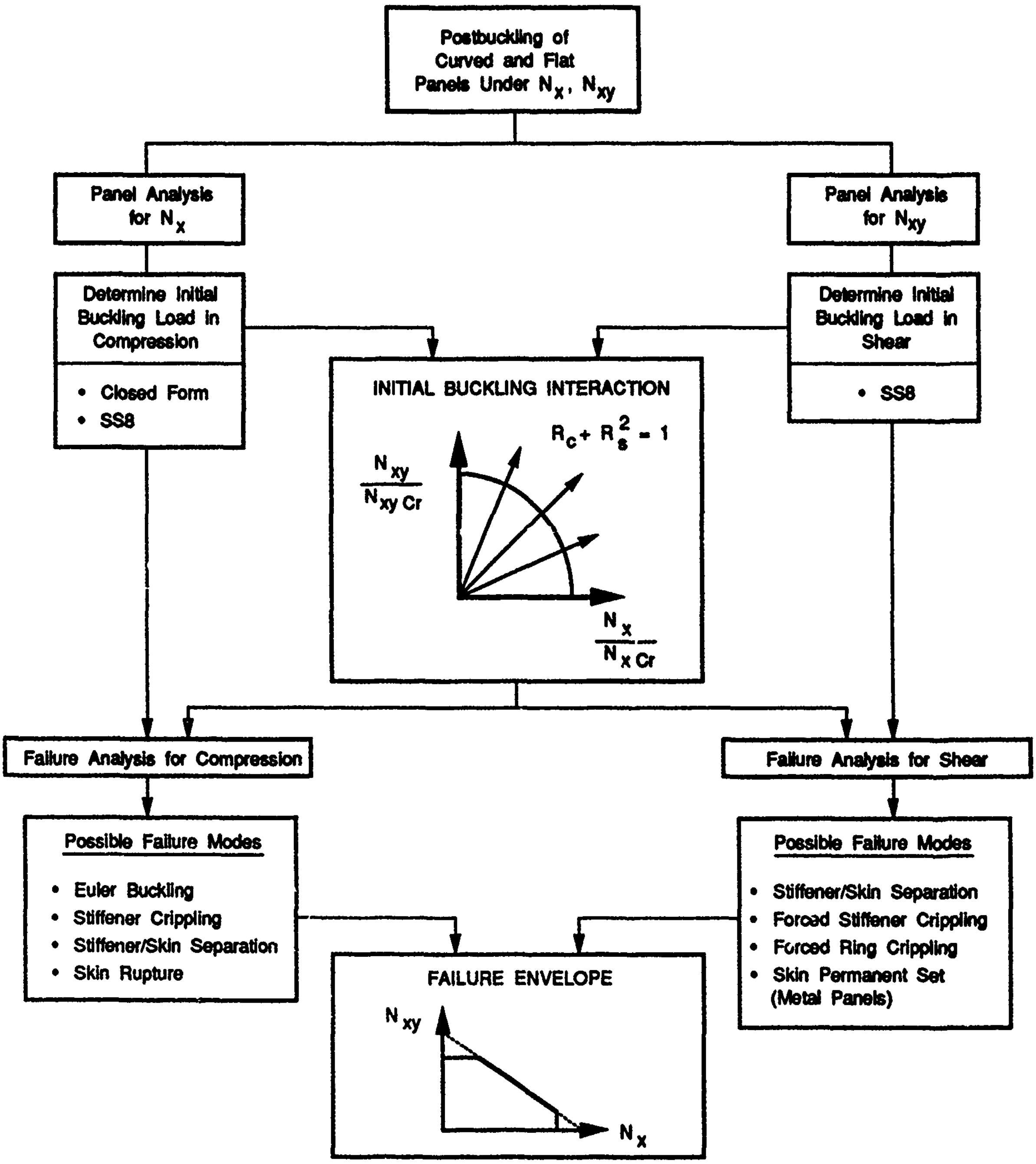


Figure 1. Semi-empirical Analysis Approach for Postbuckled Stiffened Panels Under Combined Loading.

E, ν modulus and Poisson's ratio for the sheet material

K_c buckling coefficient determined from Figure 2 (References 6 and 7)

The theoretical value of K_c is obtained from the buckling equations for thin cylindrical shells and is a function of the nondimensional curvature Z of the panel expressed as

$$Z = \frac{b_s^2 (1-\nu^2)^{1/2}}{rt_w}$$

where r is the radius of the cylindrical panel. Experimental data (Reference 7) have shown that K_c is also a function of the r/t ratio for the panel. The design curves of Figure 2, obtained from test data, show this dependence of K_c on r/t.

Compression buckling strains for curved composite panels can be accurately determined through the use of computer codes SS8 (Reference 8) and BUCLASP-2 (Reference 9), for example. However, for an approximate calculation of the skin buckling strain, the simplified equation given below can be used.

$$\epsilon_{cr}^{w} = \left(\frac{m\pi}{L}\right)^2 \frac{1}{E_{xw}t_w}\left[D_{11} + 2(D_{12}+2D_{66})\left(\frac{nL}{mb_w}\right)^2 + D_{22}\left(\frac{nL}{mb_w}\right)^4\right]$$
$$+ \frac{E_{yw}}{\left(\frac{m\pi}{L}\right)^2 R^2\left[E_{xw} - \left(2\nu_{xyw}E_{yw} - \frac{E_{xw}E_{yw}}{G_{xyw}}\right)\left(\frac{nL}{mb_w}\right)^2 + E_{yw}\left(\frac{nL}{mb_w}\right)^4\right]} \tag{2}$$

where D_{ij} are the terms of the bending stiffness matrix of the composite skin, E_{xw}, E_{yw}, G_{xyw}, ν_{xyw}, and t_w are the web elastic constants and thickness, respectively, L is the panel length, b_w is the width of the skin, R is the radius of curvature of the panel and n and m are integer coefficients representing the number of half buckle waves in the width and length direction, respectively. The lowest value of strain for various values of n and m represents the buckling strain of the specimen.

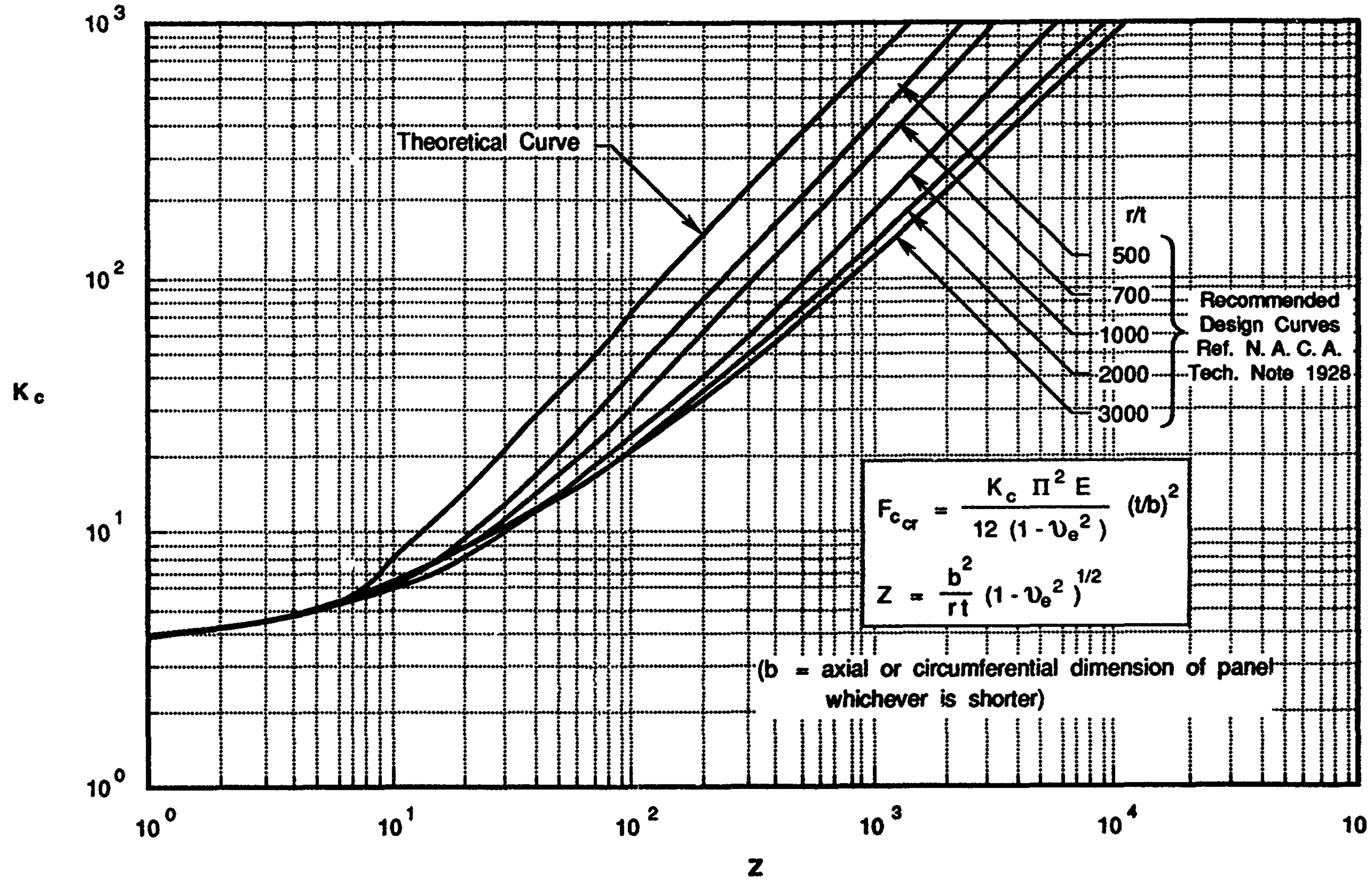


Figure 2. Axial Compression Buckling Coefficients for Long Curved Plates. (Reference 6).

The effective width of the skin, b_w, was assumed to be equal to the distance between the two adjacent stiffeners measured from one stiffener flange centroid to the next stiffener flange centroid. Note that b_w is less than the stringer spacing h_s.

Equation 2 was derived in Reference 10 from the equations developed for the buckling of orthotropic complete cylinders by making simplifying assumptions.

The shear buckling stress or strain for composite webs can be calculated using program SS8 (Reference 9). The buckling stress for curved metal webs can be calculated using:

$$\tau_{cr,elastic} = \frac{K_{s1}\pi^2 E h_s^2}{12R^2Z^2} \quad \text{if } h_r \geq h_s$$

$$= \frac{K_{s2}\pi^2 E h_s^2}{12R^2Z^2} \quad \text{if } h_s \geq h_r \qquad (3)$$

where,

K_{s1}, K_{s2} = critical shear stress coefficients for simply supported curved plates, given in Reference 6

R = panel radius, in.

E = Young's modulus for the material, psi

$$Z = \frac{h_s^2}{Rt_w}\sqrt{(1-\nu^2)} \quad \text{if } h_r \geq h_s$$

$$= \frac{h_r^2}{Rt_w}\sqrt{(1-\nu^2)} \quad \text{if } h_s \geq h_r$$

ν = Poisson's ratio for the material

The composite panel buckling loads obtained from program SS8 are in terms of running loads $N_{x,cr}$ and $N_{xy,cr}$ for compression and shear loading, respectively. The critical buckling stresses for metal panels, Equations 1 and 3 can be converted to running loads as follows:

$$\overset{o}{N}_{x_{cr}} = F_{cr} \cdot t_w$$

$$\overset{o}{N}_{xy_{cr}} = \tau_{cr,elastic} \cdot t_w$$

where t_w is the skin thickness.

For combined compression and shear loading, the buckling loads can be computed from (Reference 6):

$$R_c + R_s^2 = 1 \tag{4}$$

where, $R_c = N_{x_{cr}}/\overset{o}{N}_{x_{cr}}$ and $R_s = N_{xy_{cr}}/\overset{o}{N}_{xy_{cr}}$. $\overset{o}{N}_{x_{cr}}$ and $\overset{o}{N}_{xy_{cr}}$ are the pure compression and pure shear buckling loads, respectively, and $N_{x_{cr}}$ and $N_{xy_{cr}}$ are the buckling loads when the shear and compression loads are acting simultaneously. The presence of compression stresses reduces the shear buckling stress and vice versa.

2.2.2 Failure Analysis and Margin Computation

Failure analysis of postbuckled structures requires identification of all possible failure modes and calculating the loads corresponding to the critical failure mode. For curved panels under combined loading a failure envelope spanning the load ratio N_x/N_{xy} values of 0 (i.e., $N_x=0$, $N_{xy}\neq0$) to ∞ (i.e., $N_x\neq0$, $N_{xy}=0$) is a convenient means for identifying the critical failure mode. The procedure to develop this failure envelope is detailed in the following subsections.

2.2.2.1 Compression Loading Failure Analysis ($N_x/N_{xy}=\infty$, $N_{xy}=0$)

The analysis for failure under compression loading has been developed and documented in Reference 1. Under compression loading the possible failure modes are:

1. Euler buckling of the stiffened panel

2. Stiffener crippling

3. Stiffener/skin separation for composite panels with cocured or bonded stiffeners

4. Skin permanent set for metal panels.

Euler Buckling Strain Calculations. The Euler buckling strain for a stiffened panel is calculated by treating the panel as a wide column with the width set equal to the stiffener spacing. The critical strain is calculated using the standard column equation:

$$\epsilon_{CR} = \frac{C\pi^2 EI}{EA\ L^2} \tag{5}$$

where, EI is the equivalent bending stiffness of the panel, EA is the equivalent axial stiffness, L is the panel length, and C is the end fixity coefficient. The fixity coefficient depends upon the support conditions at the panel ends. Most compression panels are tested by flat end testing and the results obtained by using C = 4 are unconservative; therefore, a value of C = 3 is recommended. The values of C for other end conditions can be obtained from Reference 6 (Subsection A18.23).

Stiffener Crippling Strain/Stress Calculation. The crippling strength of metal stiffeners is calculated using the well established Needham or Gerard methods documented in Reference 7. In the present program, the Gerard method was used since it is a generalization of the Needham method and was derived from a broader data base. The empirical Gerard equation for calculating the crippling stress for 2 corner sections, such as the Z, J and channel sections, is:

$$\frac{F_{cs}}{F_{cy}} = 3.2\left[\left(\frac{t^2}{A}\right)\left(\frac{E}{F_{cy}}\right)^{1/3}\right]^{0.75} \tag{6}$$

where

F_{cs} = crippling stress for the section, psi

F_{cy} = compressive yield stress of the material, psi

t = element thickness, in.

A = section area, in^2

A design curve based on Equation 6 is shown in Figure 3 taken from Reference 6 Additional crippling equations that apply to sections other than 2 corner sections are also given in Reference 6.

In order to calculate the crippling strains for stiffeners made of composite materials, a semi-empirical methodology was developed in the program. The methodology consists of modeling the stiffener in terms of interconnected flat plate elements, calculating the initial buckling and crippling strains for each element, and determining the crippling strain for the stiffener as the lowest strain that causes crippling of the most critical element in the stiffener section. It should be noted here that the absolute minimum of the crippling strains for the various plate elements is not necessarily the stiffener crippling strain; element criticality with respect to stiffener stability has to be considered as well. The procedural details of this methodology given in the following paragraphs provide additional clarifications relating to the determination of the most critical plate element.

The first step in calculating the stiffener crippling strain is to model the stiffener as an interconnected assembly of plate elements. As examples, plate element models of a hat-section and a J-section stiffener are shown in Figure 4. The hat-section stiffener is made up of four elements, whereas, the J-section stiffener consists of nine elements.

The crippling strains for the plate elements are calculated from empirical equations of the form

$$\frac{\epsilon_{cs}}{\epsilon_{cr}} = \alpha \left(\frac{\epsilon_{cu}}{\epsilon_{cr}} \right)^{\beta} \tag{7}$$

where

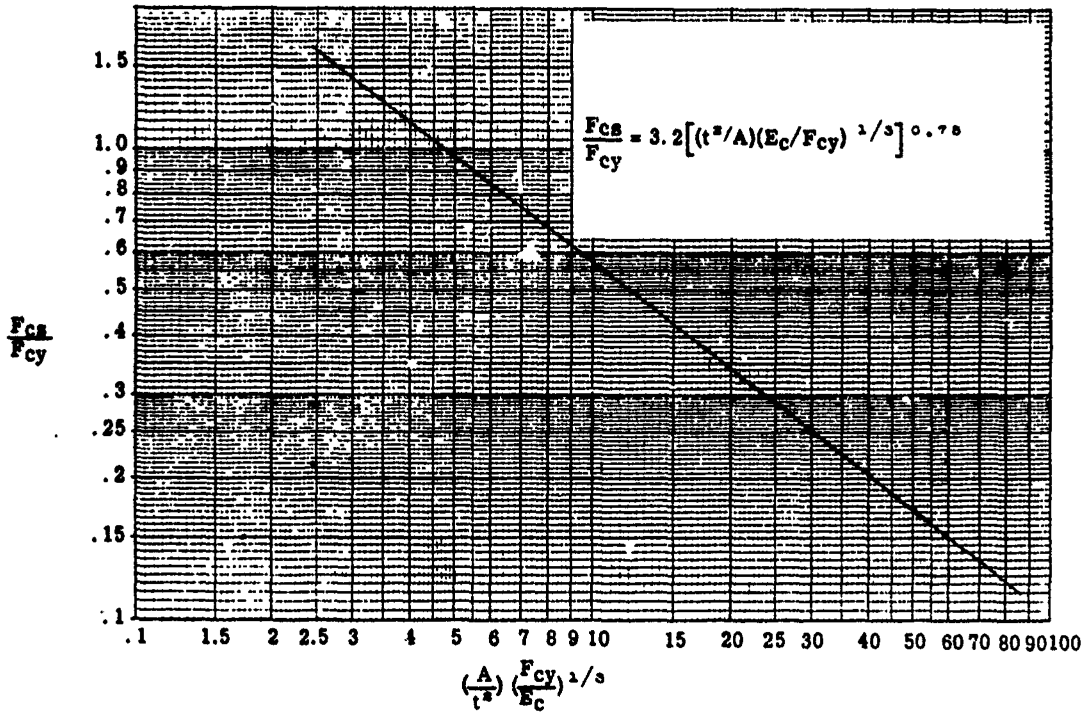


Figure 3. Crippling Stress F_{cs} for Two Corner Sections e.g., Z, J, and Channel Sections (Reference 6, Figure C7-9).

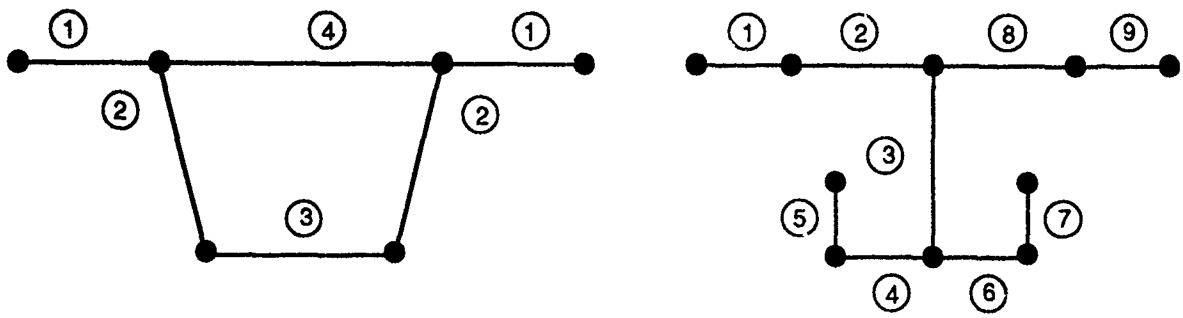


Figure 4. Plate Element Models of Hat- and J-Section Stiffeners.

ϵ_{cs} = crippling strain of the plate element

ϵ_{cr} = initial buckling strain of the plate element

ϵ_{cu} = compression ultimate strain for the plate element laminate

α,β = material dependent coefficients obtained from test data

Equation 7 has the same functional form as that used by Gerard (Reference 7) for metal stiffeners. The coefficients α and β depend on the plate edge conditions and have been obtained in References 11 and 12 from a large data base for plate elements that are connected on both sides (e.g., Elements 2, 3, and 4 of the hat-section stiffener shown in Figure 4). The crippling strain for stiffener plate elements connected on both sides is given by (Reference 12):

$$\epsilon_{cs} = 0.56867\epsilon_{cr}\left(\frac{\epsilon_{cu}}{\epsilon_{cr}}\right)^{0.47567} \tag{8}$$

where ϵ_{cr}, the buckling strain for the plate element, is given by (Reference 13):

$$\epsilon_{cr} = \frac{2\pi^2}{b^2 t\ E_x}\left(\sqrt{D_{11}D_{22}} + D_{12} + 2D_{66}\right) \tag{9}$$

In Equation 9

b = plate element width

t = plate element thickness

E_x = compression modulus of the plate laminate along the longitudinal direction

D_{ij} = terms from the laminate bending stiffness matrix, (i,j = 1, 2, 6)

Equation 9 applies to plate elements for which the length-to-width ratio (L/b, where L = stiffener length) is at least 4.

The crippling strain for plate elements that are connected on one side only is calculated using the following equation:

$$\epsilon_{cc} = 0.4498\epsilon_{cr}\left(\frac{\epsilon_{cu}}{\epsilon_{cr}}\right)^{0.72715} \tag{10}$$

where,

$$\epsilon_{cr} = \frac{12\ D_{66}}{b^2 t\ E_x} + \frac{4\pi^2 D_{11}}{L^2 t\ E_x} \tag{11}$$

L = length of the stiffener

with the other nomenclature remaining the same as for Equations 8 and 9.

The coefficients in Equation 10 were obtained by fitting Equation 7 to the crippling data generated from tests on one-edge free plates in References 11 and 12. Data for two material systems, T300/5208 and AS/3501 graphite/epoxy, were pooled to obtain Equation 10.

In Equations 8 through 11, the thickness of plate elements attached to the skin is taken as the sum of the plate element and the cocured skin thicknesses. In the case of the hat-section stiffener, crippling strains for plate elements representing the skin only, such as Element 4 in Figure 4 are also calculated. Another consideration in calculating the crippling strain for stiffener flange elements attached to the skin is the choice of an appropriate element width. For example, in most practical designs the stiffener flanges attached to the skin are tapered by dropping-off plies as shown in Figure 5 for a hat-section stiffener. The flange plate element width in this case is defined as the width to the end of the taper with the weighted average of the element thickness added on to the attached skin thickness to obtain the total thickness for use in Equations 8 through 11.

Equations 8 through 11 are quite general in nature and take into account ply composition, stacking sequence, and material characteristics. The ply composition, i.e., the percentages of 0°, 45°, and 90° plies, is reflected in the compression ultimate strain ϵ_{cu}. Stacking sequence effects are accounted for in the expression for ϵ_{cr} where the bending stiffnesses D_{ij} are used. The D_{ij}'s and ϵ_{cu} also account for mechanical property changes from one material system to another. Use of strain rather than stress for crippling calculations provides another significant advantage in that laminate nonlinearity (e.g., stress-strain response of ±45° laminates) is accounted for by way of the compression ultimate strain ϵ_{cu}.

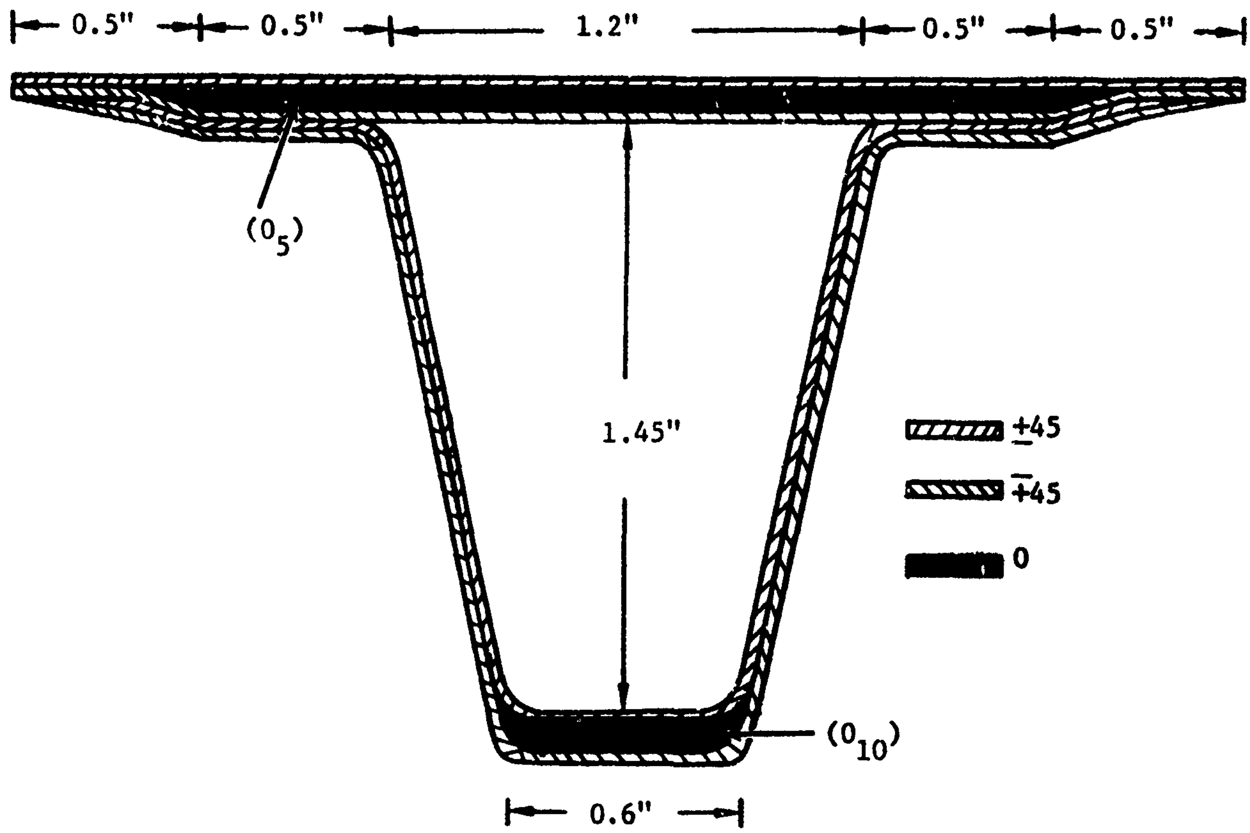


Figure 5. Ply Drop-Offs in Hat-Section Stiffener.

Failure Load Calculation. The failure load for the panel is determined as the lowest of the loads calculated for the various instability modes mentioned above, for stiffener-web separation in composite panels, and for skin or stiffener yielding in metal panels. The methods for failure load calculation are given in the following paragraphs.

Failure Load Due to Euler Buckling. The failure load due to Euler buckling is calculated using the following equation:

$$P_E = \epsilon_{cr}^{E} \left(E_{xs}A_s + E_{xw}b_w t_w \right) \tag{12}$$

where,

ϵ_{cr}^{E} = Euler buckling strain determined using Equation 5

E_{xs} = Compression modulus of the stiffener in the loading direction

A_s = Cross-sectional area of the stiffener

E_{xw} = Compression modulus of the web (skin) in the loading direction

b_w = Stiffener spacing

t_w = Skin thickness

Failure Load Due to Stiffener Crippling. In order to determine the failure load due to stiffener crippling, it is necessary to determine the load carried by the stiffener and the panel web individually. The load carried by the stiffener (P_s) is determined as follows:

1. Determine the two lowest crippling strains (ϵ_{cc1}) and (ϵ_{cc2}) of all the elements making up the cross-section using Equations 8 through 11.

2. If the element with the lowest crippling strain (ϵ_{cc1}) is normal to the axis of least bending stiffness of the cross-section, the stiffener will fail at a strain equal to ϵ_{cc1}, and the corresponding failure strain of the stiffener is given by:

$$P_S = E_{xs}A_x \, \epsilon_{cc1} \tag{13}$$

3. If the element with the lowest crippling strain is parallel to the axis of least bending stiffness of the cross-section, the stiffener will carry additional load until the second member in the cross-section becomes critical due to crippling. In this case the load carried by the stiffener is given by:

$$P_S = (EA)_1 (\epsilon_{cc1} - \epsilon_{cc2}) + \epsilon_{cc2} E_{xs} A_s \tag{14}$$

where $(EA)_1$ is the extensional stiffness of the member becoming critical first, and the stiffener failure strain

$$\epsilon_s^{cc} = \epsilon_{cc2}$$

The total load carried by the panel is the sum of the load carried by the stiffener up to crippling and the load carried by the buckled skin. In order to calculate the load carried by the skin, the effective width concept is utilized. The effective width for metal panels is calculated using the semi-empirical equation given below (Reference 6):

$$w = 1.9 t_w \sqrt{\frac{E}{F_{st}}} \tag{15}$$

where

w = effective width of the skin after initial buckling

t_w = skin thickness

F_{st} = stress in the stringer

For composite panels, in the absence of any other guidelines, Equation 15 expressed in terms of strain is used to compute the effective skin width. Thus,

$$w = 1.9 t_w (\epsilon_s)^{-0.5} \tag{15A}$$

for composite skins where ϵ_s = strain in the stiffener.

Thus, the total load carried by the panel for a stiffener crippling mode of failure is given by:

$$P_{cc} = P_s + P_w \tag{16}$$

where

P_{cc} = load carried by the panel at stiffener crippling

P_s = stiffener load given by Equation 14

P_w = load carried by the skin

The load P_w is calculated as:

$$P_w = F_{cs}\, w\, t_w = 1.9 t_w^2 \sqrt{EF_{cs}} \tag{17}$$

for metal panels, and for composite panels as:

$$P_w = 1.9 t_w^2 E_{xw} (\epsilon_s^{cc})^{0.5} \tag{18}$$

Failure Load Due to Stiffener/Web Separation. Failure of composite stiffened panels due to stiffener/web separation is a common mode of failure in the postbuckling range. It is extremely difficult to predict this failure, even by using rather sophisticated analysis methods. The attempts to date on making such predictions have been inconclusive. A simple empirical equation to predict such failure was developed in this program. The correlation of experimental data with the predicted failure loads based upon this equation is surprisingly good. The empirical equation was derived by analogy with the crippling data for plates with one edge simply supported and one edge free. It is hypothesized that when the panel web strain reaches the crippling strain the interfacial stresses become high enough to cause failure. The equation should represent the lower bound on predicted failure loads. Any attempts to improve the interface (for example, by stitching, riveting, etc.) can result in higher failure loads.

$$P_{ss} = \epsilon_{ss} (E_{xs} A_S + E_{xw} b_w t_w) \tag{19}$$

where

$$\epsilon_s^{ss} = 0.4498\, \epsilon_{cr} \left(\frac{\epsilon_{cu}}{\epsilon_{cr}} \right)^{0.72715} \tag{20}$$

ϵ_{ss} = Failure strain for stiffener/web separation

P_{ss} = Failure load for the stiffener/web separation mode

The metal compression panel analysis methodology outlined in the preceding paragraphs has been experimentally validated (e.g., Reference 7) and is representative of current usage. In the case of composite panels, available composite compression panel test data were utilized to validate the semi-empirical analysis (Reference 1).

2.2.2.2 Shear Loading Failure Analysis ($N_x/N_{xy}=0$; $N_{xy}\neq0$)

Flat or curved shear panel analysis is accomplished by means of the semi-empirical tension field theory developed by Kuhn (Reference 4) for metal panels. In Reference 1 the tension field theory was modified for application to composite shear panels by taking into account material anisotropy.

The essential elements of the generalized (for application to metals as well as composites) tension field theory and its application are summarized in Figure 6. Details of the semi-empirical analyses required to perform the various steps in Figure 6 are given in the following paragraphs. The equations as presented below pertain to cylindrically curved composite panels and to flat composite panels if terms incorporating the radius of curvature R are set equal to zero. Use of the appropriate values for elastic constants in the equations permits their direct application to metal panels. The analysis procedure is based entirely on the theory presented in Reference 4 unless specifically noted.

Computation of the Diagonal Tension Factor. The diagonal tension factor k characterizes the degree to which diagonal tension is developed in the skin of stiffened panels loaded in shear. A value of k = 0 characterizes an unbuckled skin with no diagonal tension; a value of k = 1.0 characterizes a web in pure diagonal tension. The diagonal tension factor is computed using the following expression:

$$k = \mathrm{Tanh}\left[\left(0.5 + 300\,\frac{t_w h_r}{R\,h_s}\right)\log\frac{\tau}{\tau^o_{cr}}\right] \tag{21}$$

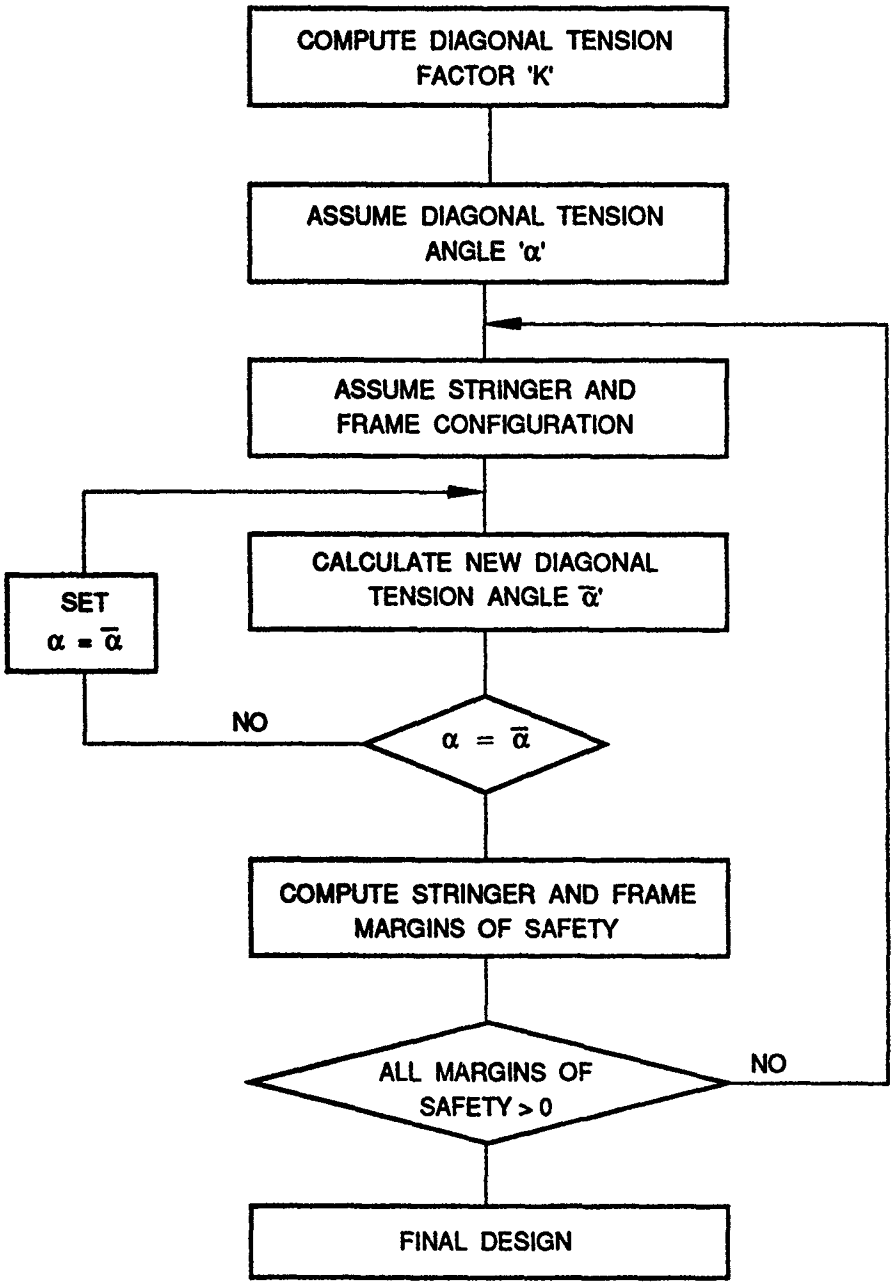


Figure 6. Application of Tension Field Theory to Shear Panels.

where

t_w = web thickness

h_r = ring spacing

h_s = stringer spacing

R = panel radius

τ = applied shear stress = N_{xy}/t_w

$\overset{o}{\tau}_{cr}$ = buckling shear stress of web under pure shear conditions

= $\overset{o}{N}_{xy_{cr}}/t_w$

The pure shear buckling stresses for composite and metal panels are calculated using the techniques given in Subsection 2.2.1.

Computation of Diagonal Tension Angle 'α'. An initial value is assigned to the diagonal tension angle 'α' that defines the angle of the 'folds' in the buckled skin. For curved web systems α=30° was found to be a convenient starting point. The actual value of α is determined by the iterative procedure outlined below.

Using the assumed initial value of α a 'new' value for α is calculated by the equation:

$$\alpha_1 = \mathrm{Tan}^{-1}\left[\frac{\epsilon - \epsilon_s}{\epsilon - \epsilon_r + R_f}\right]^{0.5} \tag{22}$$

where

$$\epsilon = \frac{\tau}{E_{w\alpha}}\left[\frac{2k}{\mathrm{Sin}2\alpha} + \frac{E_{w\alpha}}{2G_{rs}}(1-k)\ \mathrm{Sin}2\alpha\right] \tag{22a}$$

$$\epsilon_s = \frac{-k\tau\ \mathrm{Cot}\alpha}{\left[\frac{EA_s}{h_s t_w} + 0.5\ (1-k)R_s E_{ws}\right]} \tag{22b}$$

$$\epsilon_r = \frac{-k\tau\ \mathrm{Tan}\alpha}{\left[\frac{EA_r}{h_r t_w} + 0.5\ (1-k)\ E_{wr}\right]} \tag{22c}$$

$$R_f = \frac{1}{24}\left(\frac{h_s}{R}\right)^2 \qquad \text{if } h_r > h_s$$

$$= \frac{1}{8}\left(\frac{h_r}{R}\right)^2 \text{Tan}^2\alpha \qquad \text{if } h_s > h_r \tag{22d}$$

For eccentric stringers and rings

$$\overline{EA}_s = EA_s \frac{EI_s}{\overline{EI}_s}$$

$$\overline{EA}_r = EA_r \frac{EI_r}{\overline{EI}_r} \tag{22e}$$

In Equations 22, ϵ is the skin strain in the diagonal tension direction, and ϵ_s and ϵ_r are the strains in the stringer and the ring leg attached to the web averaged over their lengths, respectively. $E_{w\alpha}$, E_{ws}, and E_{wr} are the web moduli in the direction of the tension field, stringers and rings, respectively. G_{rs} is the web shear modulus. $\overline{EA}_s$ and $\overline{EA}_r$ are the effective axial stiffnesses of the stringers and the rings, respectively, calculated with respect to the skin mid-surface. EI is the bending stiffness about the stiffener neutral axis and $\overline{EI}$ the bending stiffness about the web midsurface.

In general, α_1, the new diagonal tension angle will not equal the initially assumed value of 30°. Therefore, α_1 is used as the next guess and the computations of Equation 22 are repeated until the process converges, i.e., $\alpha_{new} \approx \alpha_{old}$.

Once the diagonal tension angle has been determined with sufficient accuracy, the next step is to compute the margins of safety.

Computation of Stringer and Frame Margins of Safety. The diagonal tension angle value computed above is now substituted in Equations 22 to obtain the diagonal tension strain in the skin, the stringer strain, and the ring strain. Next, the stringer and ring strains averaged over the cross section and the length (ϵ_{ave}) and the maximum strains in the legs attached to the web (ϵ_{max}) are computed using the following equations:

$$\epsilon_{s_{ave}} = \epsilon_s \frac{\overline{EA}_s}{EA_s} \tag{23}$$

$$\epsilon_{s_{max}} = \epsilon_s \left[1 + 0.775\ (1-k)(1-0.8\frac{h_r}{h_s})\right] \qquad \text{if } h_s > h_r$$

$$= \epsilon_s \left[1 + 0.775\ (1-k)(1-0.8\frac{h_s}{h_r})\right] \qquad \text{if } h_s < h_r \tag{24}$$

$$\epsilon_{r_{ave}} = \epsilon_r \frac{\overline{EA}_r}{EA_r} \tag{25}$$

$$\epsilon_{r_{max}} = \epsilon_r \left[1 + 0.775\ (1-k)(1-0.8\frac{h_r}{h_s})\right] \qquad \text{if } h_s > h_r$$

$$= \epsilon_r \left[1 + 0.775\ (1-k)(1-0.8\frac{h_s}{h_r})\right] \qquad \text{if } h_s < h_r \tag{26}$$

The stringer and ring crippling mode of failure is then analyzed for by computing the stringer and ring forced crippling strains (ϵ_{os} and ϵ_{or}, respectively) using the following equations:

$$\epsilon_{os} = 0.00058\left[\left(\frac{\epsilon_{all}\ E_{cs}}{1000}\right)^{0.4} k^{2/3}\left(\frac{t_{us}}{t_w}\right)^{1/3}\right] \tag{27}$$

$$\epsilon_{or} = 0.00058\left[\left(\frac{\epsilon_{all}\ E_{cr}}{1000}\right)^{0.4} k^{2/3}\left(\frac{t_{ur}}{t_w}\right)^{1/3}\right] \tag{28}$$

where ϵ_{all} is the laminate allowable strain, E_{cs} and E_{cr}, are the compression modulus of the stringer and ring leg attached to the web, respectively, and t_{us} and t_{ur} are the thickness of the stringer and the ring leg attached to the web.

The critical stiffener strains corresponding to the bending stiffness required for stiffener stability are calculated using Equations 29 and 30.

$$\epsilon_{sB} = \frac{4\pi^2 EI_s}{E_{xs}A_s h_r 2} \tag{29}$$

$$\epsilon_{rB} = \frac{4\pi^2 EI_r}{E_{xr}A_r h_s 2} \tag{30}$$

where ϵ_{sB} and $_{rB}$ are the Euler buckling strains for the stiffener and the ring, respectively.

The margins of safety can now be computed for each of the possible failure modes by comparing the calculated strain values with the allowables. Thus, to ensure positive margins, the following failure modes are examined and the corresponding inequalities verified.

1. For stringer and ring stability i.e., no column failure

$$\epsilon_{sB} > \epsilon_{s_{ave}}$$

$$\epsilon_{rB} > \epsilon_{r_{ave}}$$

2. For stability of the entire panel, i.e., to prevent buckling of the web as a whole, before formation of the tension field

$$EI_s > E_s t_w^3 (\frac{3h_s}{h_r} - 2)h_s$$

$$EI_r > E_r t_w^3 (\frac{3h_r}{h_s} - 2)h_r \tag{31}$$

3. For prevention of forced crippling of stiffeners

$$\epsilon_{os} > \epsilon_{smax}$$

$$\epsilon_{or} > \epsilon_{rmax}$$

An additional check needs to be performed for metal panels where yielding or permanent set in the web is likely due to excessive skin deformation. The only available criterion for permanent set check has been empirically obtained from tests on flat aluminum metal panels. Its applicability to other materials or curved panels has not been verified. Thus, in the absence of any other guidelines, the flat panel requirement that the maximum allowable value of the diagonal tension factor k_{all} be limited to

$$k_{all} = 0.78 - (t-0.012)^{0.50} \tag{32}$$

at design ultimate load to prevent permanent buckling of the web at limit load. is used in the present analysis.

2.2.2.3 Combined Loading Failure Analysis (N_x/N_{xy}=B; $\overset{o}{N}_x cr/\overset{o}{N}_{xy} cr$=A)

The effects of shear and compression loading interaction have to be accounted for in a combined loading failure analysis. For the combined loading case, the additional considerations are:

1. The buckling stresses are reduced in accordance with the interaction given in Equation 4.
2. Compression stresses in the stiffeners prior to buckling are those due to the directly applied compression only. However, after buckling the compression stresses due to diagonal tension must be added to the direct compression.
3. The allowable stress calculation for the stiffeners must account for an interaction between the forced crippling (panel shear induced) and natural crippling (direct compression induced) modes of stiffener failure.
4. Calculation of the stiffener stresses due to applied shear loads is modified to account for the presence of the compression load.

The buckling interaction equation can be rewritten as

$$N_{xcr}/\overset{o}{N}_{xcr} + (N_{xycr}/\overset{o}{N}_{xycr})^2 = 1$$

then,

$$N_{xycr} = \overset{o}{N}_{xycr}\sqrt{1-(N_{xcr}/\overset{o}{N}_{xcr})} \qquad (33)$$

The diagonal tension factor k is expressed as

$$k = \tanh\left[\left(0.5 + 300\,\frac{t_w h_r}{R\,h_s}\right)\log\frac{N_{xy}}{N_{xycr}}\right] \qquad (34)$$

where

N_{xy} = applied shear load

N_{xycr} = shear buckling load for combined loading as calculated from Equation 33.

Calculation of k using Equation 34 is subject to the auxiliary rules that if $h_s > h_r$ then replace h_r/h_s with h_s/h_r and if the resulting ratio is greater than 2 then use a value of 2 for the ratio.

The diagonal tension angle α is computed iteratively using the same procedure as for pure shear, but with appropriate modifications to the stiffener strain expressions. Thus, if α=30° initially then the new α is calculated using Equation 22 where ϵ, the skin strain is obtained from Equation 22a, the ring strain from Equation 22c and the stiffener strain from the following expression:

$$\epsilon_s = \frac{-k\tau\cot\alpha}{\left[\frac{EA_s}{h_s t_w} + 0.5(1-k)E_{ws}R_s\right]} \tag{35}$$

where $R_c + R_s^2 = 1$.

As in the pure shear case, sufficient iterations need to be performed so that $\alpha_{new} \approx \alpha_{old}$.

Computation of Stiffener Margin of Safety. The total stiffener load can be expressed as:

$$P_s = P_x + P_{xy} \tag{36}$$

where P_x is the load in the stiffener due to direct compression and P_{xy} is the load in the stiffener due to the diagonal tension folds. The resulting stiffener strain can be expressed as (References 6 and 7)

$$\epsilon_s = \frac{-N_x h_s}{\left[(EA)_s + w\, t_w E_{ws}\right]} - \frac{k\, N_{xy} \cot\alpha}{t_w\left[\frac{(EA)_s}{h_s t_w} + 0.5(1-k)E_{ws}R_s\right]} \tag{37}$$

where the negative signs denote a compression strain, w is the effective width of the skin after buckling as obtained from Equations 17 or 18 and R_s from

Equation 4. The average and maximum strains in the stiffener can be computed by analogy to the pure shear case, i.e.,

$$\epsilon_{s_{ave}} = \frac{-N_x h_s}{\left[(EA)_s + w\, t_w E_{ws}\right]} - \frac{k\, N_{xy} \cot\alpha}{t_w\left[\frac{(EA)_s}{h_s t_w} + 0.5(1-k)E_{ws}R_s\right]} \cdot \frac{(\overline{EA})_s}{(EA)_s} \tag{38}$$

$$\epsilon_{s_{max}} = \frac{-N_x h_s}{\left[(EA)_s + w\, t_w E_{ws}\right]} - \frac{k\, N_{xy} \cot\alpha}{t_w\left[\frac{(EA)_s}{h_s t_w} + 0.5(1-k)E_{ws}R_s\right]} \cdot D_o \tag{39}$$

where

$$\begin{aligned} D_o &= \left[1 + 0.775(1-k)(1-0.8\,\frac{h_r}{h_s})\right] && \text{if } h_s > h_r \\ &= \left[1 + 0.775(1-k)(1-0.8\,\frac{h_s}{h_r})\right] && \text{if } h_s < h_r \end{aligned} \tag{40}$$

In computing margins of safety for stiffener design, the above strains have to be compared against the Euler buckling strain and the stiffener crippling allowable strain. For Euler buckling, it is immaterial whether the stiffener compressive strain arises from the direct compression load, P_x, or from the diagonal tension action caused by P_{xy}. Euler buckling failure is assumed to take place when $\epsilon_{s_{ave}}$ given by Equation 38 above reaches ϵ_{SB} given by Equation 29. The nature of stiffener crippling under combined loading, however, requires that the interaction between the strain due to direct compression and the strain due to diagonal tension be accounted for. This is because crippling under diagonal tension is caused by forced deformation of the stiffener leg attached to the web, whereas direct compression causes crippling failure by local instability of the entire stiffener section. An empirical expression for this interaction has been given in Reference 4 for curved metal panels. For generic application to metal and composite panels the

Reference 4 interaction is expressed in terms of strains as follows:

$$\frac{\epsilon_s^{co}}{\epsilon_s^{cc}} + \left(\frac{\epsilon_s^{so}}{\epsilon_{os}}\right)^{1.5} \leq 1.0 \qquad (41)$$

where

ϵ_s^{co} = the direct compression strain

ϵ_s^{so} = the compression strain due to diagonal tension which cause stiffener crippling while acting simultaneously

ϵ_s^{cc} = the stiffener crippling strain under pure compression loading as computed from Equations 9 and 10, and the procedure given in Section 2.2.2.

ϵ_{os} = the forced crippling strain of the stiffener under pure shear loading calculated from Equation 27.

The margin of safety is computed as follows:

$$M.S. = \frac{1}{\left[\frac{\epsilon_s^c}{\epsilon_{cc}^s} + \left(\frac{\epsilon_{smax}^s}{\epsilon_{os}}\right)^{1.5}\right]} - 1$$

where

$$\epsilon_s^c = \frac{-N_x h_s}{[(EA)_s + w\ t_w E_{ws}]}$$

and

$$\epsilon_{s_{max}}^s = \frac{-kN_{xy} \cot\alpha}{t_w\left[\frac{(EA)_s}{h_s t_w} + 0.5(1-k)E_{ws}R_s\right]} \cdot D_o \qquad (42)$$

<u>Computation of Ring Margin of Safety</u>. Metal panel test data show that the hoop compression stresses in the ring due to diagonal tension are unaffected by the axial compression on the curved panel as a whole. Therefore, Equations 22c, 25, and 26 can be readily used to compute the ring strains and margins of safety.

2.2.3 Automated Semi-Empirical Design Methodology

The design procedure outlined above has been coded in a computer program called PBUKL for use as a design tool. Detailed documentation of this program is given in Reference 14. The program is an extension of TENWEB, works interactively, and has several built-in stiffener profiles for design flexibility.

Program PBUKL was used to design the curved panels tested in this study.

SECTION 3

ENERGY METHOD BASED ANALYSIS DEVELOPMENT

3.1 PROBLEM FORMULATION

The energy approach was used for problem formulation. The problem was formulated for a stiffened, curved anisotropic laminated plate. The laminate was assumed to be balanced and symmetric. A small imperfection in the lateral displacement was also included in the formulation. The panel geometry and the coordinate system are shown in Figure 7. This figure also shows the relationship between the overall postbuckling structural configuration and the panel geometry used in the analysis. Since the adjacent bays are assumed to deform in an identical fashion, a single bay was analyzed. Figure 7 shows that the material properties for the skin are A^*_{ij} and D^*_{ij}, where A^*_{ij} (A^*_{11}, A^*_{12}, A^*_{22} and A^*_{66}) is the skin stiffness matrix and D^*_{ij} (D^*_{11}, D^*_{12}, D^*_{16}, D^*_{22}, D^*_{26} and D^*_{66}) is the skin rigidity matrix. The material axes 1 and 2 are assumed to coincide with the panel geometry coordinate axes x and y, respectively. The panel, with length a and width b, is bounded by stringers along the straight edges and frames or rings along the curved edges. The cross-sectional area, Young's modulus and moment of inertia of the stringers are A_s, E_s and I_s respectively. Those of the frames are A_f, E_f and I_f. The radius of the curved panel is R.

The energy expressions are written in terms of the displacement components u, v and w in the x, y and z directions, respectively. The panel is assumed fixed along x = o and subjected to a system of combined compression (N_x) and shear (N_{xy}) load along the edge x = a. The boundary conditions for the displacement components are therefore given by:

$$
\begin{aligned}
&u = v = w = o && \text{for } x = o \\
&w = o && \text{for } x = a \qquad (43)\\
&w = o && \text{for } y = o \text{ and } y = b
\end{aligned}
$$

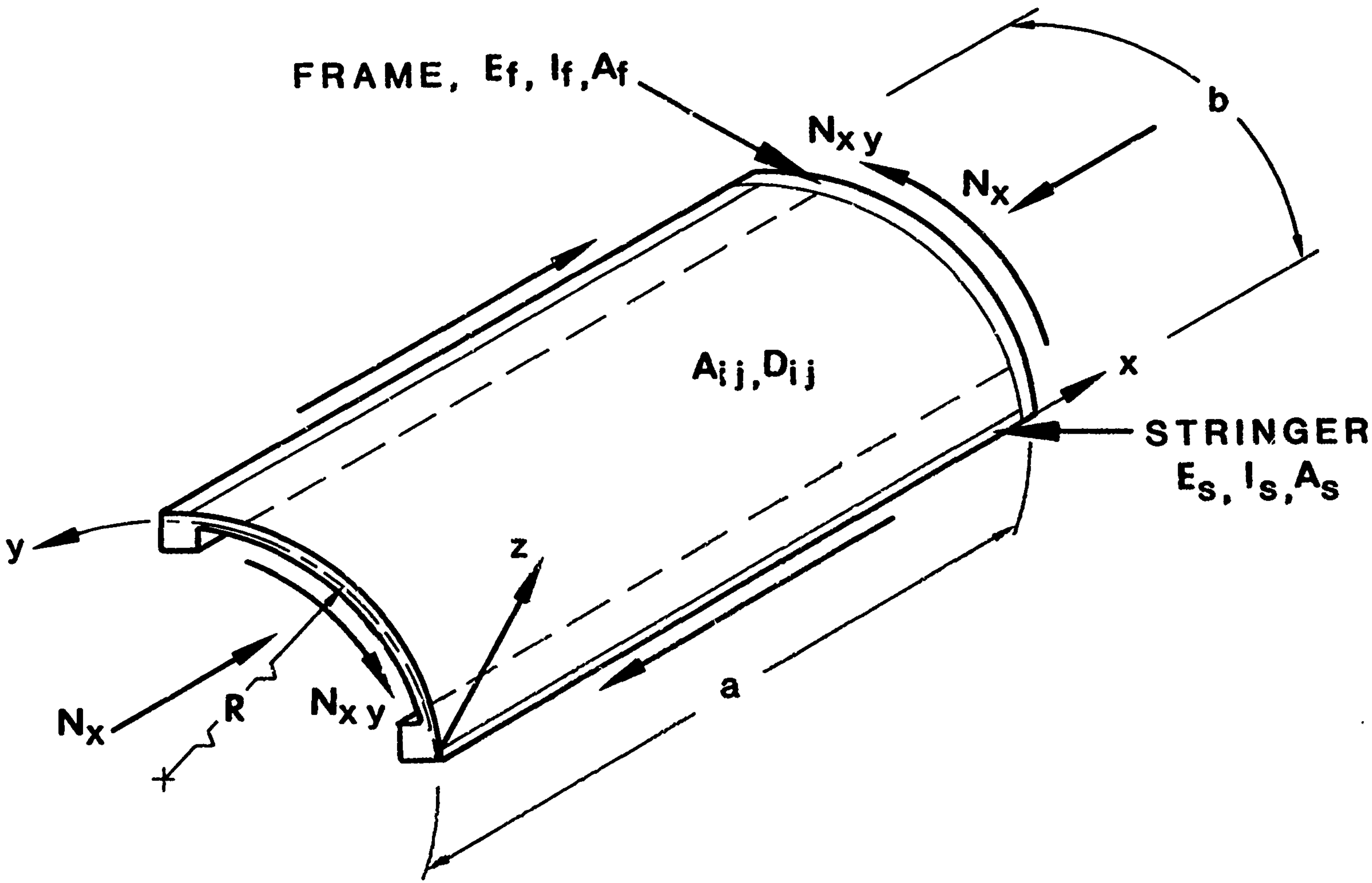


Figure 7. Curved Panel Geometry and Coordinate System.

The total potential energy, π, is the sum of the strain energy stored in the skin, U_w, in the stringers, U_s, in the frames, U_f, and the potential of the external load, Ω, and is written as:

$$\pi = U_w + U_s + U_f + \Omega \tag{44}$$

The strain energy in the skin for an anisotropic plate with $A^*_{16} = A^*_{26} = 0$ is given by

$$\begin{aligned} U_w = \frac{1}{2} \int_v & \{A^*_{11} \varepsilon_x^2 + 2A^*_{12}\varepsilon_x\varepsilon_y + A^*_{22}\varepsilon_y^2 + A^*_{66}\gamma_{xy}^2 \\ & + D^*_{11}w_{,xx}^2 + 2D^*_{12}w_{,xx}w_{,yy} + 4D^*_{16}w_{,xx}w_{,xy} \\ & + D^*_{22}w_{,yy}^2 + 4D^*_{26}w_{,yy}w_{,xy} + 4D^*_{66}w_{,xy}^2\}\, dv \end{aligned} \tag{45}$$

where ε_x, ε_y and γ_{xy} are the strain components. Commas denote differentiation with respect to the subscripted variables.

The strains are expressed in terms of the displacements u, v and w using the nonlinear strain-displacement relations:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\frac{\partial w}{\partial x}\right]^2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\frac{\partial w}{\partial y}\right]^2 + \frac{w}{r} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \end{aligned} \tag{46}$$

In the derivations that follow, the coordinate variables x and y are normalized with respect to their respective panel dimensions. The normalized coordinates (ξ,η) are given by

$$\xi = \frac{x}{a} \quad , \quad \eta = \frac{y}{b} \tag{47}$$

The strain displacement relations can then be rewritten as:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{a} u_\xi + \frac{1}{2a^2} w_\xi^2 \\
\varepsilon_y &= \frac{1}{b} v_n + \frac{1}{2b^2} w_\eta^2 + \frac{w}{R} \\
\gamma_{xy} &= \frac{1}{a} v_\xi + \frac{1}{b} u_\eta + \frac{1}{ab} w_\xi w_\eta
\end{aligned} \tag{48}$$

In Equation (48) and hereafter, the subscripts to the displacements u, v and w denote differentiation with respect to the subscript variables.

Substituting the strain-displacement relations, Equation (48), into Equation (45) the strain energy stored in the skin becomes:

$$\begin{aligned}
U_w = \frac{ab}{2} \Bigg\{ & \int_o^1 \int_o^1 \overset{*}{A}_{11} \left[\frac{1}{a^2} u_\xi^2 + \frac{1}{a^3} u_\xi w_\xi^2 + \frac{1}{4a^4} w_\xi^4 \right] d_\xi d_\eta \\
& + \int_o^1 \int_o^1 \overset{*}{A}_{12} \left[\frac{2}{ab} u_\xi v_\eta + \frac{2}{aR} u_\xi w + \frac{1}{ab^2} u_\xi w_\eta^2 \right. \\
& \qquad \left. + \frac{1}{a^2 b} v_\eta w_\xi^2 + \frac{1}{a^2 R} w w_\xi^2 + \frac{1}{2a^2 b^2} w_\xi^2 w_\eta^2 \right] d_\xi d_\eta \\
& + \int_o^1 \int_o^1 \overset{*}{A}_{22} \left[\frac{1}{b^2} v_\eta^2 + \frac{2}{bR} v_\eta w + \frac{1}{R^2} w^2 + \frac{1}{b^3} v_\eta w_\eta^2 \right. \\
& \qquad \left. + \frac{1}{b^2 R} w w_\eta^2 + \frac{1}{4b^4} w_\eta^2 \right] d_\xi d_\eta
\end{aligned} \tag{49}$$

$$+ \int_0^1 \int_0^1 A_{66}^* \left[\frac{1}{b^2} u_\eta^2 + \frac{1}{a^2} v_\xi^2 + \frac{2}{ab} u_\eta v_\xi + \frac{2}{ab^2} u_\eta w_\xi w_\eta \right.$$

$$\left. + \frac{2}{a^2 b} v_\xi w_\xi w_\eta + \frac{1}{a^2 b^2} w_\xi^2 w_\eta^2 \right] d\xi d\eta$$

$$+ \int_0^1 \int_0^1 \left[\frac{D_{11}^*}{a^4} w_{\xi\xi}^2 + \frac{2D_{12}^*}{a^2 b^2} w_{\xi\xi} w_{\eta\eta} + \frac{4D_{16}^*}{a^3 b} w_{\xi\xi} w_{\xi\eta} \right.$$

$$\left. \left. + \frac{D_{22}^*}{b^4} w_{\eta\eta}^2 + \frac{4D_{26}^*}{ab^3} w_{\eta\eta} w_{\xi\eta} + \frac{4D_{66}^*}{a^2 b^2} w_{\xi\eta}^2 \right] d\xi d\eta \right\}$$

The strain energy in the stringer is

$$U_s = \frac{A_s E_s}{2a} \int_0^1 u_\xi^2 \; (\xi, 0) \; d\xi + \frac{I_s E_s}{2a^3} \int_0^1 v_{\xi\xi}^2 \; (\xi, 0) \; d\xi \tag{50}$$

The strain energy in the frame is

$$U_F = \frac{A_f E_f}{2b} \int_0^1 v_\eta^2 \; (1, \eta) \; d\eta + \frac{I_f E_f}{2b^3} \int_0^1 u_{\eta\eta}^2 \; (1, \eta) \; d\eta \tag{51}$$

The potential of the external loads is

$$\Omega = - bN_{xx} \int_0^1 u(1, \eta) \; d\eta - bN_{xy} \int_0^1 v(1, \eta) \; d\eta \tag{52}$$

The solution method employs the principle of minimum potential energy. In applying the principle of minimum potential energy, the displacement components are assumed to be functions of the independent variables ξ and η. The selected functions must satisfy the displacement boundary conditions given in Equation (43) and minimize the total potential energy. A generalized series expression for the displacement functions with unknown coefficients was selected for the present analysis and these are as follows:

$$
\begin{aligned}
u &= A_{nm}f_1 + a_1 a\xi \\
v &= B_{nm}f_2 + b_1 a\xi \qquad\qquad (53) \\
w &= c_{nm}f_3 + D_{nm}f_4 + w_0 f_5,
\end{aligned}
$$

In Equation (53), the functions $f_i = f_i(x,y;n,m)$, $i = 1, 2, 3, 4$, are arbitrarily selected admissible functions. The expressions such as $A_{nm}f_i$ are shorthand expression for a double series, i.e.,

$$
A_{nm}f_i = \sum_{n=1}^{N} \sum_{m=1}^{M} A_{nm}\, f_1\, (x,y;n,m)
$$

The coefficients A_{nm}, B_{nm}, C_{nm}, D_{nm}, a_1 and b_1 are unknown coefficients to be determined by minimizing the total potential energy. The term W_o in Equation (53) is the initial imperfection at the panel center. The function $f_5 = f_5(\xi, \eta)$ is the initial imperfection function in terms of the lateral displacement and satisfies the displacement boundary conditions.

Substituting Equation (53) into the energy expressions and using the definitions given in Appendix A for the individual energy integrals, the energy expressions finally become:

$$
\begin{aligned}
U_w = \overset{*}{A}_{11} \Bigg\{ & \frac{b}{2a} A_{nm} A_{pq} G^{11}_{11} + b a_1 A_{nm} F^{1}_{1} + \frac{ab}{2} a_1^{2} \\
& + \frac{b}{2a^2} \left(A_{nm} C_{pq} C_{rs} H^{111}_{133} + 2\, A_{nm} C_{pq} D_{rs} H^{111}_{134} + A_{nm} D_{pq} D_{rs} H^{111}_{144}\right) \\
& + \frac{bw_o}{a^2} \left(A_{nm} C_{pq} H^{111}_{135} + A_{nm} D_{pq} H^{111}_{145}\right) + \frac{b}{2a} \Big(a_1 C_{nm} C_{pq} G^{11}_{33} \\
& + 2a_1 C_{nm} D_{pq} G^{11}_{34} + a_1 D_{nm} D_{pq} G^{11}_{44}\Big) + \frac{bw_o}{a} \left(a_1 C_{nm} G^{11}_{35} + a_1 D_{nm} G^{11}_{45}\right) \\
& + \frac{bw_o^2}{2a^2} A_{nm} H^{111}_{155} + \frac{bw_o^2}{2a} a_1 G^{11}_{55} + \frac{b}{8a^3} \Big[C_{nm} C_{pq} C_{rs} C_{tu} I^{1111}_{3333} \\
& + 4\, C_{nm} C_{pq} C_{rs} D_{tu} I^{1111}_{3334} + 6\, C_{nm} C_{pq} D_{rs} D_{tu} I^{1111}_{3344} \qquad (54)\ \text{(Cont'd)}
\end{aligned}
$$

$$+ 4\ C_{nm}\ D_{pq}\ D_{rs}\ D_{tu}\ I^{1111}_{3444} + D_{nm}\ D_{pq}\ D_{rs}\ D_{tu}\ I^{1111}_{4444}$$

$$+ 4\ w_o\ (C_{nm}\ C_{pq}\ C_{rs}\ I^{1111}_{3335} + 3\ C_{nm}\ C_{pq}\ D_{rs}\ I^{1111}_{3345}$$

$$+ 3\ C_{nm}\ D_{pq}\ D_{rs}\ I^{1111}_{3445} + D_{nm}\ D_{pq}\ D_{rs}\ I^{1111}_{4445}) + 6\ w_o\ (C_{nm}\ C_{pq}\ I^{1111}_{3355}$$

$$+ 2\ C_{nm}\ D_{pq}\ I^{1111}_{3455} + D_{nm}\ D_{pq}\ I^{1111}_{4455}) + 4\ w_o\ (C_{nm}\ I^{1111}_{3555}$$

$$\left. \left. + D_{nm}\ I^{1111}_{4555}) + w_o\ I^{1111}_{5555} \right] \right\}$$

$$+ A^{*}_{12} \left\{ A_{nm}\ B_{pq}\ G^{12}_{12} - aa_1\ B_{nm}\ F^{2}_{2} \right.$$

$$+ \frac{b}{R}\ (A_{nm}\ C_{pq}\ G^{10}_{13} + A_{nm}\ D_{pq} G^{10}_{14} + w_o\ A_{nm}\ G^{10}_{15})$$

$$+ \frac{ab}{R}\ (a_1\ C_{nm}\ F^{o}_{3} + a_1\ D_{nm}\ F^{o}_{4} + w_o\ a_1\ F^{o}_{5})$$

$$+ \frac{1}{2b}\ (A_{nm}\ C_{pq}\ C_{rs}\ H^{122}_{133} + 2A_{nm}\ C_{pq}\ D_{rs}\ H^{122}_{134} + A_{nm}\ D_{pq}\ D_{rs}\ H^{122}_{144})$$

$$+ \frac{w_o}{b}\ (A_{nm}\ C_{pq}\ H^{122}_{135} + A_{nm}\ D_{pq}\ H^{122}_{145}) + \frac{a}{2b}\ (a_1\ C_{nm}\ C_{pq}\ G^{22}_{33}$$

$$+ 2a_1\ C_{nm}\ D_{pq}\ G^{22}_{34} + a_1\ D_{nm}\ D_{pq}\ G^{22}_{44}) + \frac{aw_o}{b}\ (a_1\ C_{nm}\ G^{22}_{35}$$

$$+ a_1\ D_{nm}\ G^{22}_{45}) + \frac{w_o^2}{2b}\ A_{nm}\ H^{122}_{155} + \frac{aw_o^2}{2b}\ a_1\ G^{22}_{55}$$

$$+ \frac{1}{2a}\ (B_{nm}\ C_{pq}\ C_{rs}\ H^{211}_{233} + 2B_{nm}\ C_{pq}\ D_{rs}\ H^{211}_{234} + B_{nm}\ D_{pq}\ D_{rs}\ H^{211}_{244}$$

$$+ \frac{w_o}{a}\ (B_{nm}\ C_{pq}\ H^{211}_{235} + B_{nm}\ D_{pq}\ H^{211}_{245}) + \frac{w_o^2}{2a}\ B_{nm}\ H^{211}_{255}$$

$$+ \frac{b}{2aR}\ [C_{nm}\ C_{pq}\ C_{rs}\ H^{011}_{333} + C_{nm}\ C_{pq}\ D_{rs}\ (2H^{011}_{334} + H^{110}_{334}) \qquad \text{(54 cont'd)}$$

$$+ C_{nm} D_{pq} D_{rs} (H^{011}_{344} + 2H^{110}_{344}) + D_{nm} D_{pq} D_{rs} H^{011}_{444}$$

$$+ w_o C_{nm} C_{pq} (2H^{011}_{335} + H^{110}_{335}) + 2w_o C_{nm} D_{pq} (H^{011}_{345} + H^{101}_{345} + {}^{110}_{345})$$

$$+ w_o D_{nm} D_{pq} (2H^{011}_{445} + H^{110}_{445}) + w_o C_{nm}(H^{011}_{355} + 2H^{101}_{355})$$

$$+ w_o^2 D_{nm} (H^{011}_{455} + 2H^{110}_{455}) + w_o^3 H^{011}_{555}]$$

$$+ \frac{1}{4ab} \Big[C_{nm} C_{pq} C_{rs} C_{tu} I^{1122}_{3333} + 2C_{nm} C_{pq} C_{rs} D_{tu} (I^{1122}_{3334} + I^{1221}_{3334})$$

$$+ C_{nm} C_{pq} D_{rs} D_{tu} (I^{1122}_{3344} + 4I^{1212}_{3344} + I^{2211}_{3344})$$

$$+ 2C_{nm} D_{pq} D_{rs} D_{tu} (I^{1122}_{3444} + I^{2112}_{3444}) + D_{nm} D_{pq} D_{rs} D_{tu} I^{1122}_{4444}$$

$$+ 2w_o C_{nm} C_{pq} C_{rs} (I^{1122}_{3335} + I^{1221}_{3335}) + 2w_o C_{nm} C_{pq} D_{rs} (I^{1122}_{3345}$$

$$+ 2I^{1212}_{3345} + 2I^{1221}_{3345} + I^{2211}_{3345}) + 2w_o C_{nm} D_{pq} D_{rs} (2I^{1122}_{3445} + I^{2112}_{3445}$$

$$+ I^{1221}_{3445} + 2I^{2211}_{3445}) + 2w_o D_{nm} D_{pq} D_{rs} (I^{1122}_{4445} + I^{1221}_{4445})$$

$$+ w_o C_{nm} C_{pq} (I^{1122}_{3355} + 4I^{1212}_{3355} + I^{2211}_{3355}) + 2w_o C_{nm} D_{pq} (I^{1122}_{3455}$$

$$+ 2I^{1212}_{3455} + 2I^{2112}_{3455} + I^{2211}_{3455}) + w_o^2 D_{nm} D_{pq} (I^{1122}_{4455} + 4I^{1212}_{4455} + I^{2211}_{4455}$$

$$+ 2w_o^3 C_{nm} (I^{1122}_{3555} + I^{2112}_{3555}) + 2w_o^3 D_{nm} (I^{1122}_{4555} + I^{2112}_{4555}) + w_o^4 I^{1122}_{5555} \Big]\Big\}$$

$$+ A^{*}_{22} \Big\{ \frac{a}{2b} B_{nm} B_{pq} G^{22}_{22} + \frac{a}{R} (B_{nm} C_{pq} G^{20}_{23} + B_{nm} D_{pq} G^{20}_{24}$$

$$+ w_o B_{nm} G^{20}_{25}) + \frac{ab}{2R^2} (C_{nm} C_{pq} G^{00}_{33} + 2C_{nm} D_{pq} G^{00}_{34} + D_{nm} D_{pq} G^{00}_{44})$$

$$+ \frac{abw_o}{R^2} (C_{nm} G^{00}_{35} + D_{nm} G^{00}_{45}) + \frac{abw_o^2}{2b^2} G^{00}_{55} + \frac{a}{2b^2} (B_{nm} C_{pq} C_{rs} H^{222}_{233} \qquad (54\ \text{Cont'd})$$

$$+ 2B_{nm} C_{pq} D_{rs} H^{222}_{234} + B_{nm} D_{pq} D_{rs} H^{222}_{244}) + \frac{aw_o}{b^2} (B_{nm} C_{pq} H^{222}_{235}$$

$$+ B_{nm} D_{pq} H^{222}_{245}) + \frac{aw_o^2}{2b2} B_{nm} H^{222}_{255} + \frac{a}{2b^R} \Big[C_{nm} C_{pq} C_{rs} H^{022}_{333}$$

$$+ C_{nm} C_{pq} D_{rs} (2H^{022}_{334} + H^{220}_{334}) + C_{nm} D_{pq} D_{rs} (H^{022}_{344} + 2H^{202}_{344})$$

$$+ D_{nm} D_{pq} D_{rs} H^{022}_{444} \Big] + \frac{aw_o}{2bR} \Big[C_{nm} \; C_{pq} \; (2H_{335} + 2H_{335} + H_{335})$$

$$+ 2C_{nm} D_{pq} (H^{022}_{345} + H^{202}_{345} + H^{220}_{345}) + D_{nm} D_{pq} (2H^{022}_{445} + H^{220}_{445}) \Big]$$

$$+ \frac{aw_o^2}{2bR} \Big[C_{nm} (H^{022}_{355} + 2H^{202}_{355}) + D_{nm} (H^{022}_{455} + 2H^{202}_{455}) \Big] + \frac{aw_o^3}{2bR} H^{022}_{555}$$

$$+ \frac{a}{8b^3} \Big[C_{nm} C_{pq} C_{rs} C_{tu} I^{2222}_{3333} + 4C_{nm} C_{pq} C_{rs} D_{tu} I^{2222}_{3334}$$

$$+ 6C_{nm} C_{pq} D_{rs} D_{tu} I^{2222}_{3344} + 4C_{nm} D_{pq} D_{rs} D_{tu} I^{2222}_{3444}$$

$$+ D_{nm} D_{pq} D_{rs} D_{tu} I^{2222}_{4444} + 4w_o (C_{nm} C_{pq} C_{rs} I^{2222}_{3335}$$

$$+ 3C_{nm} C_{pq} D_{rs} I^{2222}_{3345} + 3C_{nm} D_{pq} D_{rs} I^{2222}_{3445} + D_{nm} D_{pq} D_{rs} I^{2222}_{4445}$$

$$+ 6w_o^2 (C_{nm} C_{pq} I^{2222}_{3355} + 2C_{nm} D_{pq} I^{2222}_{3455} + D_{nm} D_{pq} I^{2222}_{4455})$$

$$+ 4w_o^3 (C_{nm} I^{2222}_{3555} + D_{nm} I^{2222}_{4555}) + w_o I^{2222}_{5555} \Big] \Big\}$$

$$+ A^*_{66} \Big\{ \frac{a}{2b} A_{nm} A_{pq} G^{22}_{11} + \frac{b}{2a} B_{nm} B_{pq} G^{11}_{22} + bb_1 B_{nm} F^1_2 + \frac{ab}{2} b_1^2$$

$$+ A_{nm} B_{pq} G^{21}_{12} + ab_1 A_{nm} F^2_1 + \frac{1}{b} \Big[A_{nm} C_{pq} C_{rs} H^{212}_{133} \qquad \text{(54 Cont'd)}$$

$$
\begin{aligned}
&+ A_{nm} C_{pq} D_{rs} (H_{134}^{212} + H_{134}^{221}) + A_{nm} D_{pq} D_{rs} H_{144}^{212} \\
&+ w_o A_{nm} C_{pq} (H_{135}^{221} + H_{135}^{212} + w_o A_{nm} D_{pq} (H_{145}^{212} + H_{145}^{221}) \\
&+ w_o A_{nm} H_{155} \Big] + \frac{1}{a} \Big[B_{nm} C_{pq} C_{rs} H_{233}^{112} + B_{nm} C_{pq} D_{rs} (H_{234}^{112} + H_{234}^{121}) \\
&+ B_{nm} D_{pq} D_{rs} H_{244}^{112} + w_o B_{nm} C_{pq} (H_{235}^{121} + H_{235}^{112}) \\
&+ w_o B_{nm} D_{pq} (H_{245}^{112} + H_{245}^{121}) + w_o^2 B_{nm} H_{255}^{112} \Big] + b_1 C_{nm} C_{pq} G_{33}^{12} \\
&+ b_1 C_{nm} D_{pq} (G_{34}^{12}) + b_1 D_{nm} D_{pq} G_{44}^{12} \\
&+ w_o \Big[b_1 C_{nm} (G_{35}^{21}) + G_{35}^{12}) + b_1 D_{nm}(G_{45}^{21} + G_{45}^{12}) \Big] + w_o^2 b_1 G_{55}^{12} \\
&+ \frac{1}{2ab} \Big[C_{nm} C_{pq} C_{rs} C_{tu} I_{3333}^{1122} + 2C_{nm} C_{pq} C_{rs} Dd_{tu} II_{3334}^{1122} + I_{3334}^{1221}) \\
&+ C_{nm} C_{pq} D_{rs} D_{tu} (I_{3344}^{1122} + 4I_{3344}^{1212} + I_{3344}^{2211}) \\
&+ 2C_{nm} D_{pq} D_{rs} D_{tu} (I_{3444}^{1122} + I_{3444}^{2112}) + D_{nm} D_{pq} D_{rs} D_{tu} I_{4444}^{1122} \\
&+ 2w_o C_{nm} C_{pq} C_{rs} (I_{3335}^{1122} + I_{3335}^{1221}) \\
&+ 2w_o C_{nm} C_{pq} D_{rs} (I_{3345}^{1122} + 2I_{3345}^{1212} + 2I_{3345}^{1221} + I_{3345}^{2211}) \\
&+ 2w_o C_{nm} Dd_{pq} D_{rs} (2I_{3445}^{1122} + I_{3445}^{2112} + I_{3445}^{1221} + 2I_{3445}^{2211} \\
&+ 2w_o D_{nm} D_{pq} D_{rs} (I_{4445}^{1122} + I_{4445}^{1221}) \\
&+ w_o^2 C_{nm} C_{pq} (I_{3355}^{1122} + 4I_{3355}^{1212} + I_{3355}^{2211}) \\
&+ 2w_o^2 C_{nm} D_{pq} (I_{3455}^{1122} + 2I_{3455}^{1212} + 2I_{3455}^{2112} + I_{3455}^{2211}) \qquad \text{(54 Cont'd)} \\
&+ w_o^2 D_{nm} D_{pq} (I_{4455}^{1122} + 4I_{4455}^{1212} + I_{4455}^{2211}) + 2w_o^3 C_{nm} (I_{3555}^{1122} + I_{3555}^{2112}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2w_o^3 D_{nm} (I^{1122}_{4555} + I^{2112}_{4555} + w_o^4 I^{1122}_{5555}] \Big\} \\
&+ \frac{bD^*_{11}}{2a^3} (C_{nm} C_{pq} G^{33}_{33} + 2C_{nm} D_{pq} G^{33}_{34} + D_{nm} D_{pq} G^{33}_{44} + 2w_o C_{nm} G^{33}_{35} \\
&+ 2w_o D_{nm} G^{33}_{45} + w_o^2 G^{33}_{55}) + \frac{D^*_{12}}{ab} \Big[C_{nm} C_{pq} G^{34}_{33} \\
&+ C_{nm} D_{pq} (G^{34}_{34} + G^{43}_{34}) + D_{nm} D_{pq} G^{34}_{44} + w_o C_{nm} (G^{43}_{35} + G^{34}_{35}) \\
&+ w_o D_{nm} (G^{34}_{45} + G^{43}_{45}) + w_o^2 G^{34}_{55} \Big] + \frac{2D^*_{16}}{a^2} \Big[C_{nm} C_{pq} G^{35}_{33} \\
&+ C_{nm} D_{pq} (G^{35}_{34} + G^{53}_{34}) + D_{nm} D_{pq} G^{35}_{44} + w_o C_{nm} (G^{35}_{35} + G^{53}_{35}) \\
&+ w_o D_{nm} (G^{35}_{45} + G^{53}_{45}) + w_o^2 G^{35}_{55} \Big] + \frac{aD^*_{22}}{2b^3} \Big[C_{nm} C_{pq} G^{44}_{33} \\
&+ 2C_{nm} D_{pq} G^{44}_{34} + D_{nm} D_{pq} G^{44}_{44} + 2w_o C_{nm} G^{44}_{35} + 2w_o D_{nm} G^{44}_{45} \\
&+ w_o G^{44}_{55} \Big] + \frac{2D^*_{26}}{b^2} \Big[C_{nm} C_{pq} G^{45}_{33} + C_{nm} D_{pq} (G^{45}_{34} + G^{54}_{34}) \\
&+ D_{nm} D_{pq} G^{45}_{44} + w_o C_{nm} (G^{45}_{35} + G^{54}_{35}) + w_o D_{nm} (G^{45}_{45} + G^{54}_{45}) \\
&+ w_o^2 G^{45}_{55} \Big] + \frac{2D^*_{66}}{ab} \Big[C_{nm} C_{pq} G^{55}_{33} + 2C_{nm} D_{pq} G^{55}_{34} \\
&+ D_{nm} D_{dpq} G^{55}_{44} + 2w_o C_{nm} G^{55}_{35} + 2w_o D_{nm} G^{55}_{45} + w_o G^{55}_{55} \Big]
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
U_s = \frac{A_s E_s}{2a} \Big[A_{nm} A_{pq} K^{11}_{11} + 2a\, a_1 A_{nm} J^{1}_{1} + a_1^2 a^2 \Big] \\
+ \frac{I_s E_s}{2a^3} B_{nm} B_{pq} K^{33}_{22}
\end{aligned}
\tag{55}
$$

$$
U_f = \frac{A_f E_f}{2b} B_{nm} B_{pq} K^{22}_{22} + \frac{I_f E_f}{2b^3} A_{nm} A_{pq} K^{44}_{11} \tag{56}
$$

$$\Omega = - N_{xx} \left[A_{nm} J_1 + a_1 a \right] b$$

$$- N_{xy} \left[B_{nm} J_2 + b_1 a \right] b \qquad (57)$$

The total potential energy is minimized with respect to the unknown coefficients. The minimization process yields a system of nonlinear algebraic equations. Details of these algebraic equations are given in Appendix A. These equations can be expressed in the following form:

$$\left[\frac{\partial \pi}{\partial A} \Big|_L \right] = \left[\frac{\partial \pi}{\partial A} \Big|_N - C \right] \qquad (58)$$

where the subscript L denotes the linear terms of the partial derivative of the total potential energy with respect to a particular unknown coefficient (A), subscript N denotes the nonlinear terms and C represents the terms that are independent of the unknown coefficients.

An alternate approach to minimize the total potential energy is to directly evaluate the energy expressions given by Equations (54) through (57). In this approach, the values of the unknown coefficients are initially assumed. The value of the total potential energy is then directly evaluated using the assumed values of the coefficients. The values of the coefficients are systematically varied and the value of the corresponding total potential energy is evaluated iteratively until the minimum potential energy is approximately achieved. The values of the unknown coefficients are then substituted into Equation (58) to verify the numerical accuracy. This procedure is adopted as an alternate approach because of a convergence problem encountered in solving the nonlinear system, Equation 58, directly.

The numerical solution procedures are discussed in Sections 3.2 and 3.3. Section 3.2 details a single mode solution, in which only one buckling mode is selected for analysis, i.e., only one term for each of the displacement functions f_i, $i = 1, 2, 3, 4$, is used. In this case, the system of Equations 58 is reduced to six equations and direct solution of Equation 58

presents little difficulty. The multi-mode solution is discussed in Section 3.3. Because of the interaction between buckling modes, numerical solution of Equation 58 is not always possible. A combination of the two approaches discussed earlier is used to determine the unknown coefficients.

In the actual numerical solution, the displacement functions have to be specifically defined. The functions that satisfy the boundary conditions and describe the displacement behavior observed in postbuckled panel tests are as follows:

$$
\begin{aligned}
f_1(\xi, \eta; n, m) &= \sin \Pi\xi \cos (n\Pi\xi - m\Pi\eta) \\
f_2(\xi, \eta; n, m) &= (1 - \cos \Pi\xi) \cos (n\Pi\xi - m\Pi\eta) \\
f_3(\xi, \eta; n, m) &= \sin n\Pi\xi \sin m\Pi\eta \\
f_4(\xi, \eta; n, m) &= \sin \Pi\xi \sin \Pi\eta \sin (n\Pi\xi - m\Pi\eta) \\
f_5(\xi, \eta) &= \sin \Pi\xi \sin \Pi\eta
\end{aligned}
\tag{59}
$$

The individual energy integrals written in terms of the above functions are evaluated in closed form. Detailed expressions of all the energy integrals are given in Appendix A.

3.2 SINGLE MODE ANALYSIS

The number of nonlinear algebraic equations in the system given by Equations (58) depend on the number of buckling modes used in the analysis. The number of equations can be calculated from the relation 4NM+2, where N is the number of buckling modes in the x-direction and M is the number of buckling modes in the y-direction. As N and M increase, the number of nonlinear terms on the right hand side of Equation 58 also significantly increases (see Equations A-9 through A-14 in Appendix A). A large number of nonlinear terms present numerical difficulty in solving Equation (58). On the other hand, although the postbuckling behavior of a stiffened panel is mixed-mode behavior in general, the displacement response is dominated by a single buckling mode. Therefore, if the dominant buckling mode is known, the postbuckling behavior can be accurately described using a single-mode analysis.

Let n be the selected buckling mode in the x-direction and m the mode in the y-direction. Then the number of equations in the nonlinear system, Equation (58) is reduced to six. Since there is no mode interaction, the energy integrals G, H, I, K in Equations A-9 through A-14 become

$$
\begin{aligned}
G^{\alpha\beta}_{ij,\,nmpq} &= G^{\alpha\beta}_{ij,\,nmnm} \\
H^{\alpha\beta\gamma}_{ijk,nmpqrs} &= H^{\alpha\beta\gamma}_{ijk,nmnmnm} \\
I^{\alpha\beta\gamma\delta}_{ijkl,\,nmpqrstu} &= I^{\alpha\beta\gamma\delta}_{ijkl,\,nmnmnmnm} \\
K^{\beta\beta}_{\alpha\alpha,nmpq} &= K^{\beta\beta}_{\alpha\alpha,\,nmnm}
\end{aligned}
\qquad (60)
$$

In addition, for the case of no initial imperfection, i.e., $w_o = o$, the total number of integrals involved in Equations 58 is reduced to 108. The reduced system of nonlinear equations can be solved with -- :y high accuracy by an iterative technique using the method of successive linearization. In this method, each of the unknown coefficient, A_{nm}, B_{nm}, C_{nm}, D_{nm}, a_1 and b_1 is assigned an initial value and substituted into the right-hand-side of Equation 58. Equation 58 now becomes a system of linear algebraic equations and can be easily solved for the new values of the unknown coefficients. Using the new set of coefficients as initial values, another set of improved coefficients can be obtained by solving the linearized system. This procedure is continued until the solution converges within a desired limit. In the actual solution, only the initial values of the coefficients at the first load level need to be assigned. At higher load levels, the initial values are obtained by extrapolating the converged solutions of the preceding load levels to reduce the number of iterations.

The results of the single-mode solution for an example problem are presented below. The panel geometry and the material properties used in this problem are:

Panel Length a = 17.5 in.

Panel Width b = 10.0 in.

Panel Radius R = 45.0 in.

$A_{11}^* = 562$ kips/in $\quad$ $A_{12}^* = 174.7$ kips/in

$A_{22}^* = 582$ kips/in $\quad$ $A_{66}^* = 225$ kips/in

$D_{11}^* = 121$ lb/in $\quad$ $D_{12}^* = 36.4$ lb/in

$D_{16}^* = D_{26}^* = 0$ $\quad$ $D_{22}^* = 121$ lb/in $\quad$ $D_{66}^* = 46.87$ lb/in

$A_s = 0.11\ in^2$ $\quad$ $E_s = 10 \times 10^6$ psi $\quad$ $I_s = 0.00615\ in^4$

$A_f = 0.5\ in^2$ $\quad$ $E_f = 10 \times 10^6$ psi $\quad$ $I_f = 5.0\ in^4$

In this example, the frame is assumed to be very rigid, this is simulated by using a relatively large moment of inertia ($5.0\ in^4$) as compared to that of the stringer ($0.00615\ in^4$). The ratio of the compression to shear load (N_x/N_{xy}) used in the example is -1.0.

The results of the single mode analysis for this example are illustrated in Figures 8 through 15. These results include solutions for $m = 1$ with n rarging from 1 to 6. The value of a_1 as a function of the total compression load ($P_{xx} = bN_x$) is shown in Figure 8. The parameter a_1 is an in-plane displacement parameter. The value of a_1 is essentially the axial strain in the x-direction due to end-shortening and is obtained from the first of Equations 53. As shown in Figure 8, a_1 varies linearly with applied load for $n = 1$. For $n \geq 2$, the end-shortening parameter becomes approximately bilinear with applied load. The load level where the slope of the end-shortening curve changes signifies initial buckling of the skin. The figure shows that the initial load is lower for some of the higher buckling modes (larger value of n). The initial buckling load for $n = 2$ is approximately 5000 lbs. and reduces to 2800 lbs for $n = 6$. These results indicate that with $m = 1$, the first two buckling modes ($n = 1$ and $n = 2$) are not likely to dominate the postbuckling behavior of the panel. This conclusion is more readily evident from the results of other parameters discussed below.

Figure 9 shows the relationship between the shear-displacement parameter b_1 and the total applied compression. The parameter b_1 is the dominant term in the y-direction displacement (shear-displacement). The value of b_1a is the average shear-displacement at the loading end of the panel ($x = a$). The results shown in Figure 9 indicate that the value of b_1

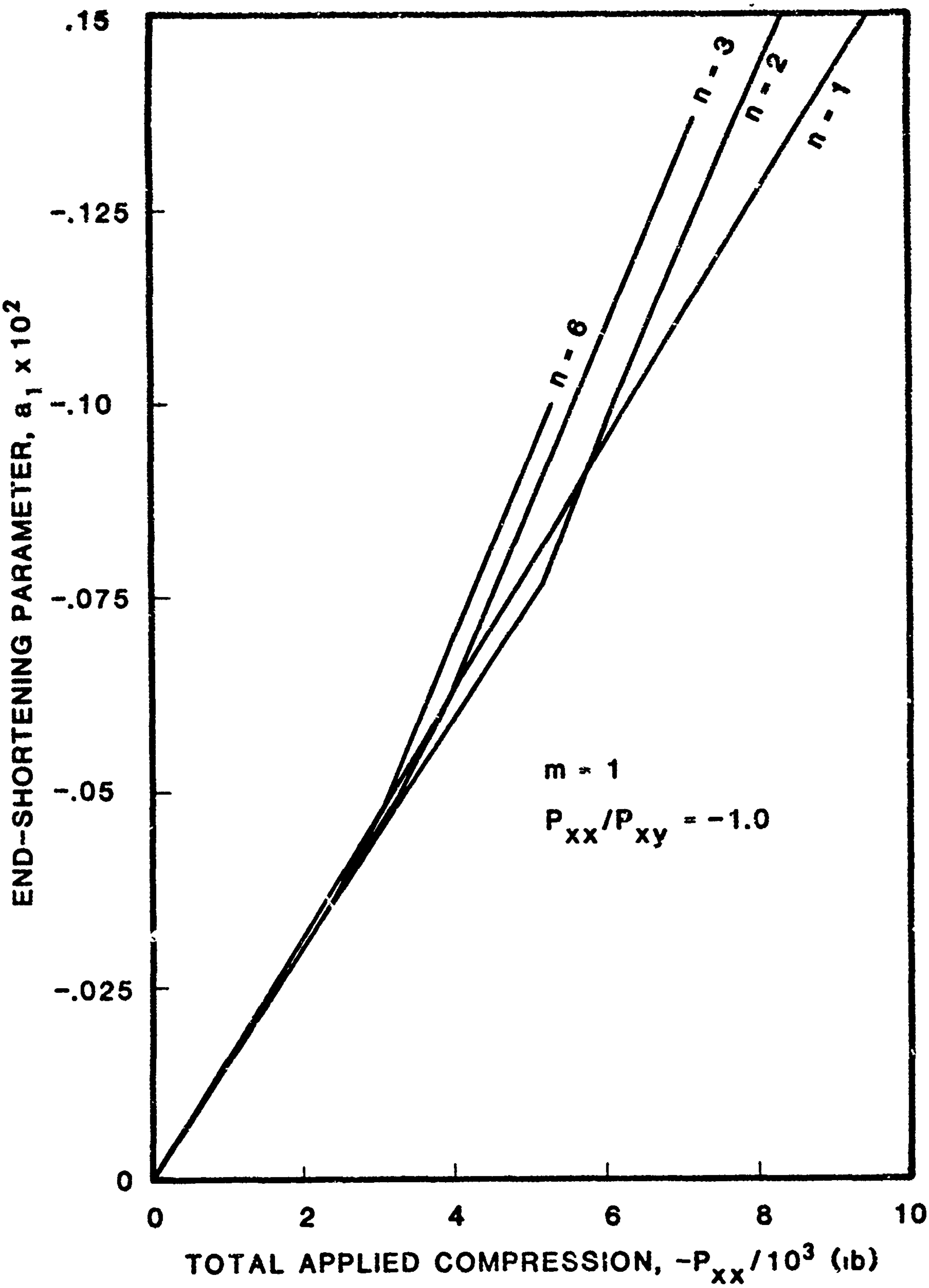


Figure 8. End Shortening Parameter as a Function of the Total Applied Compression Load.

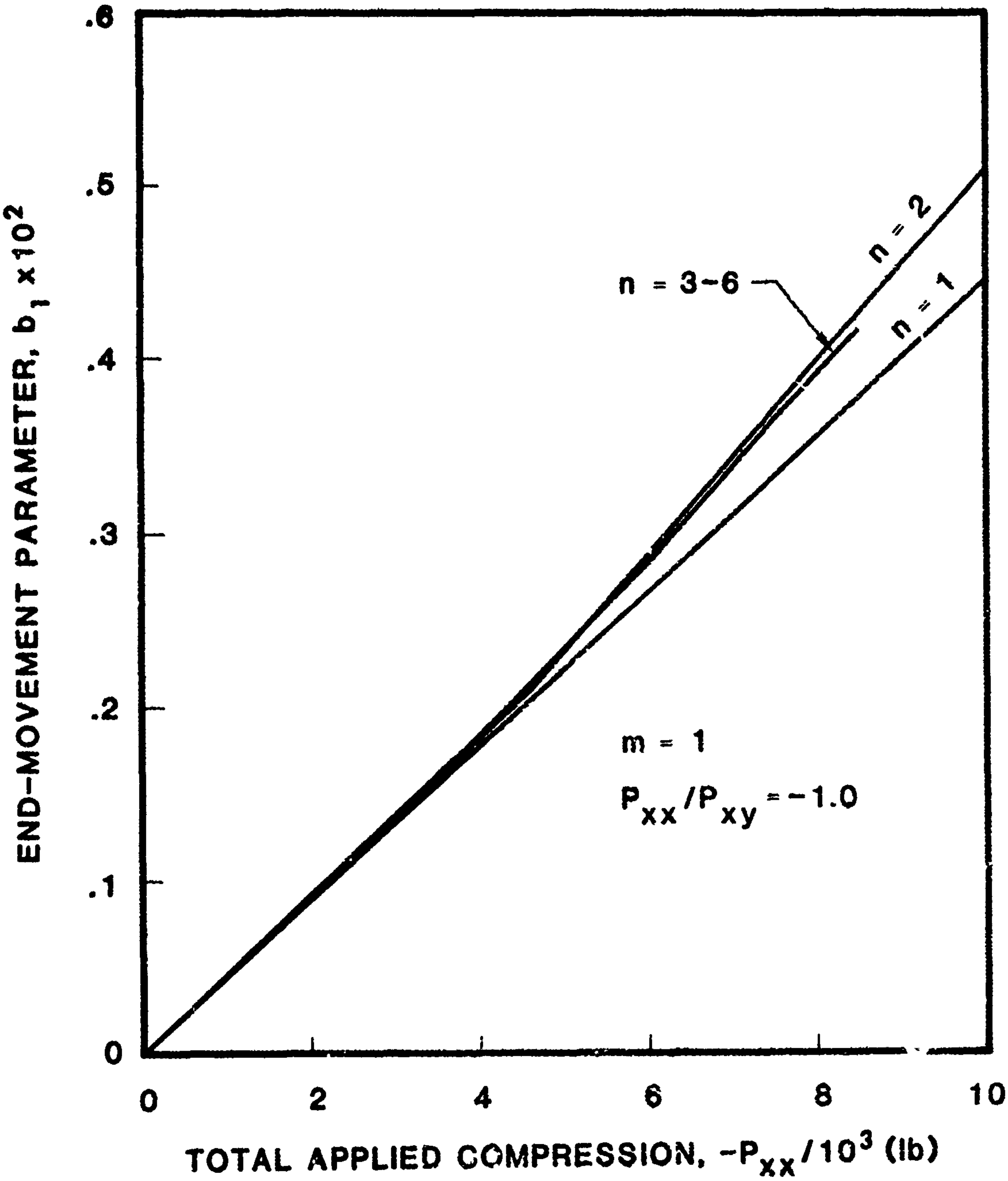


Figure 9. Shear End-Displacement Parameter b_1 as a Function of the Applied Compression Load.

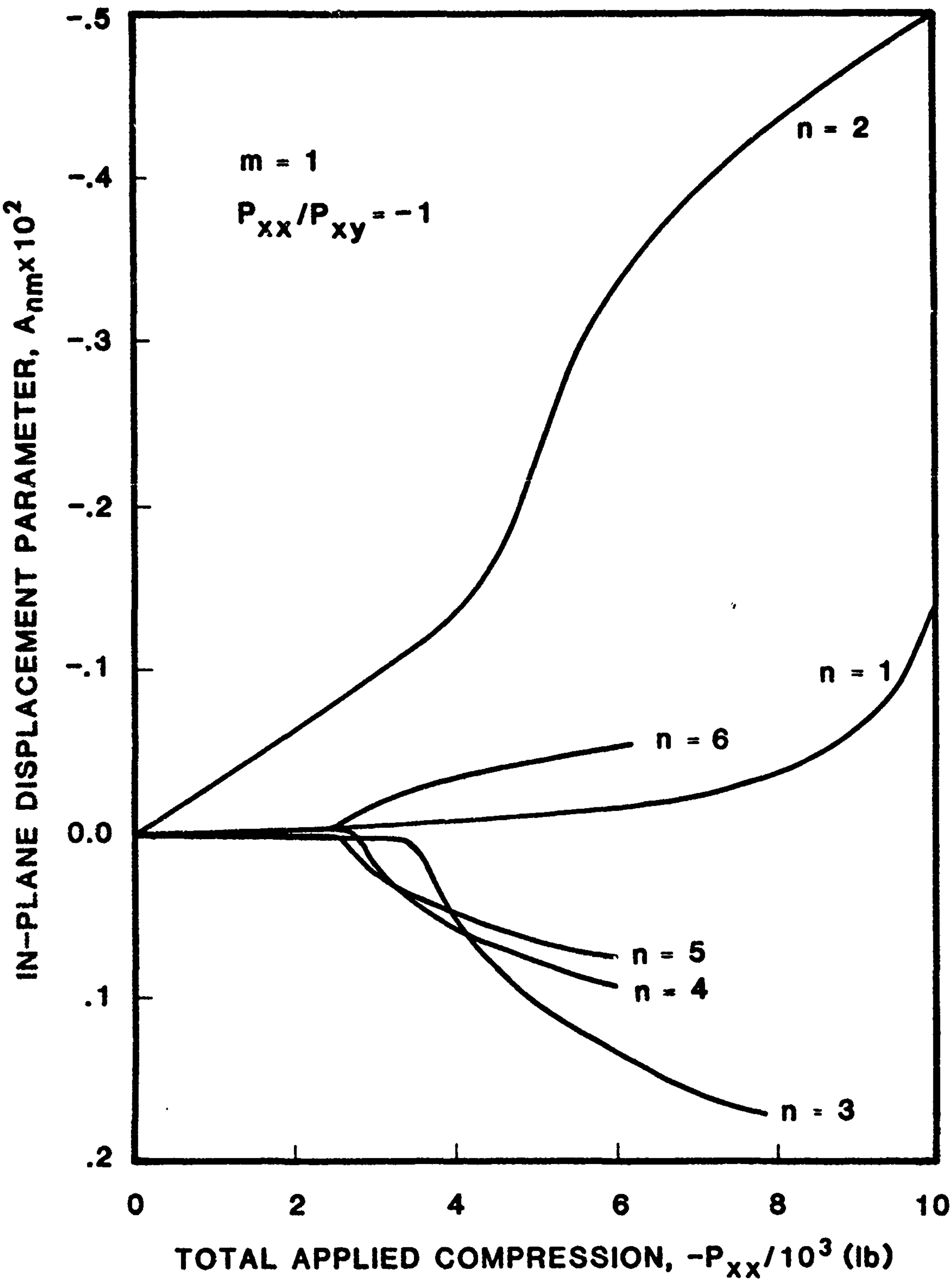


Figure 10. Displacement Coefficient A_{nm} as a Function of the Applied Compression Load.

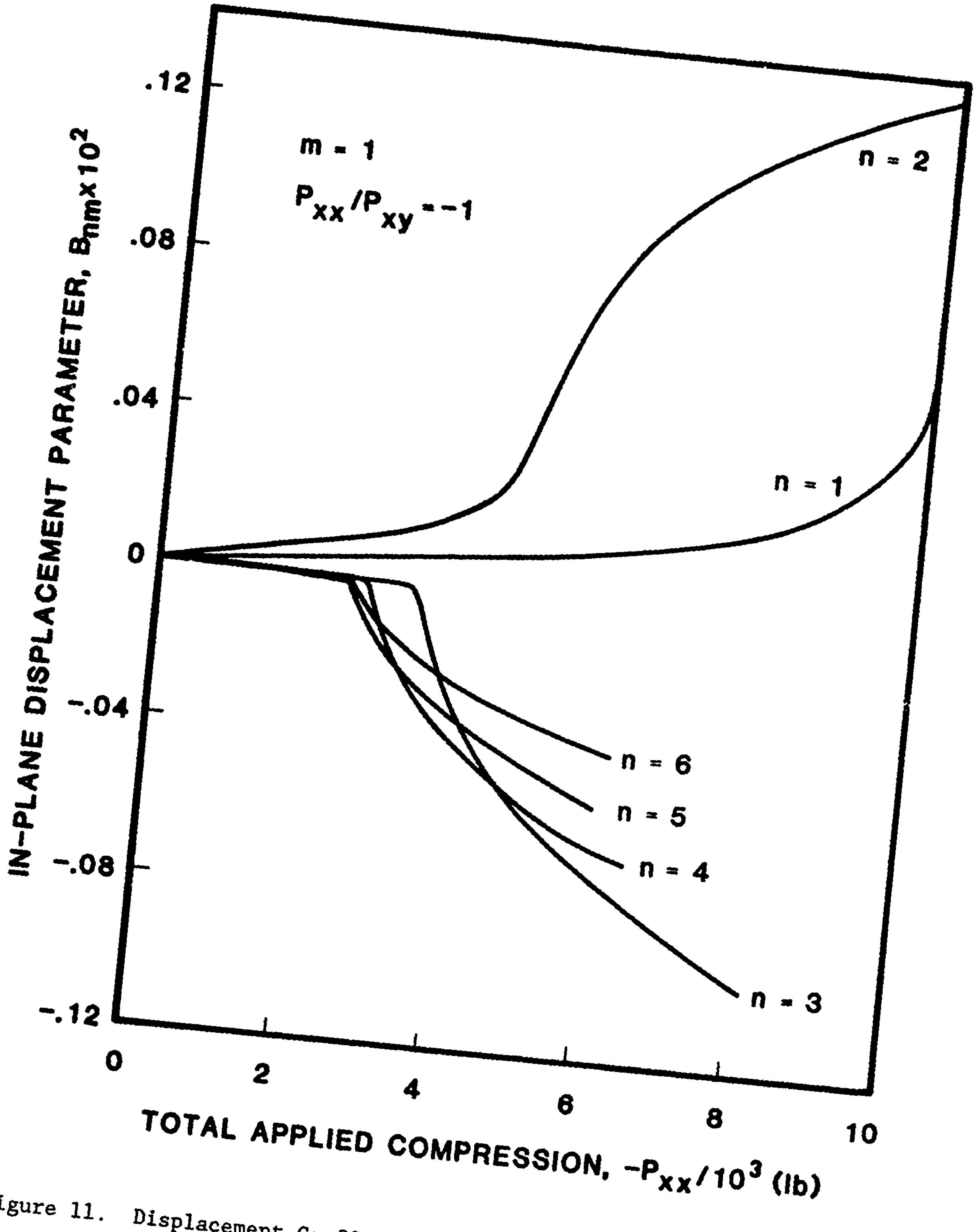


Figure 11. Displacement Coefficient B_{nm} as a Function of the Applied Compression Load.

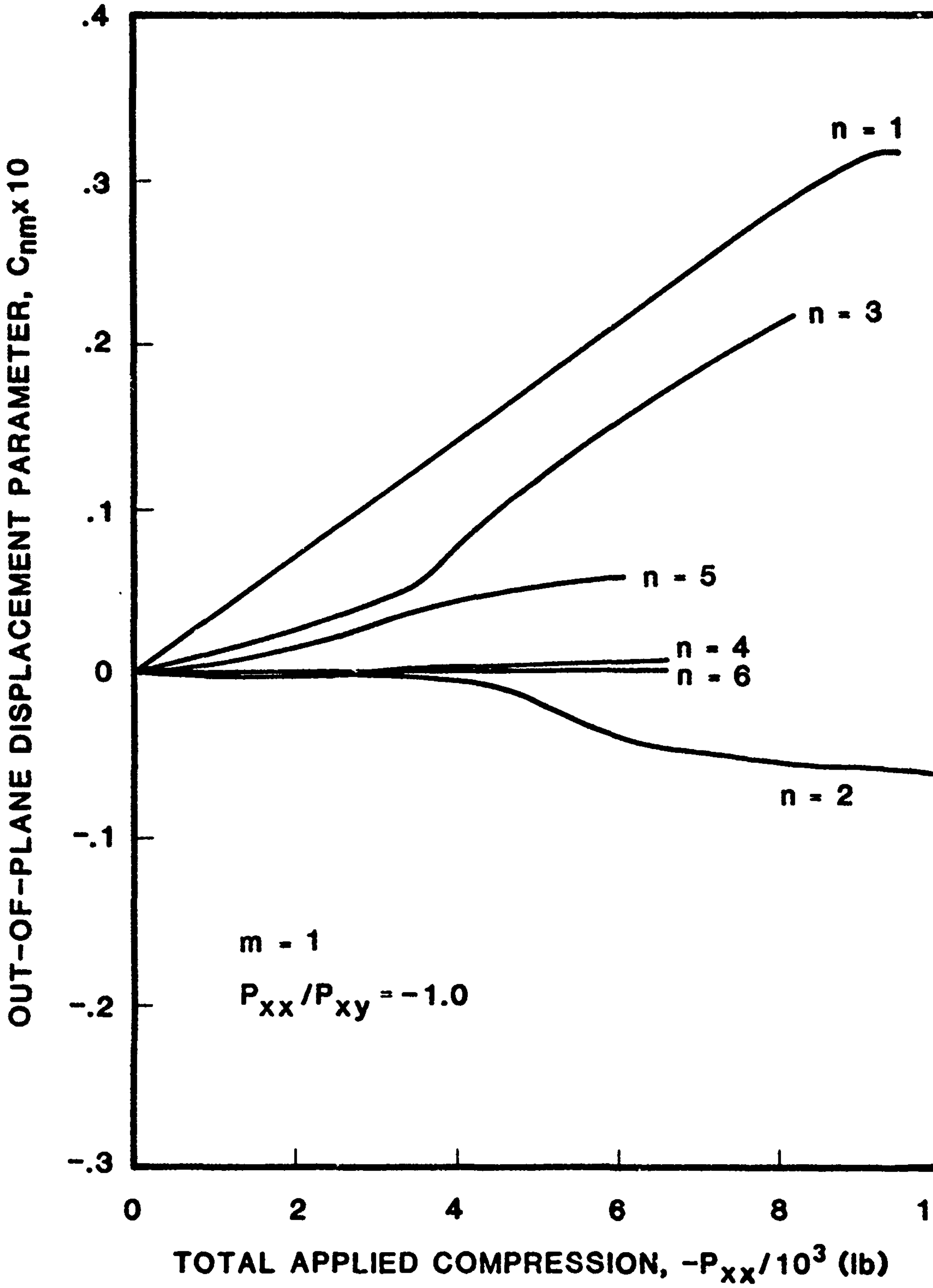


Figure 12. Displacement Coefficient C_{nm} as a Function of the Applied Compression Load.

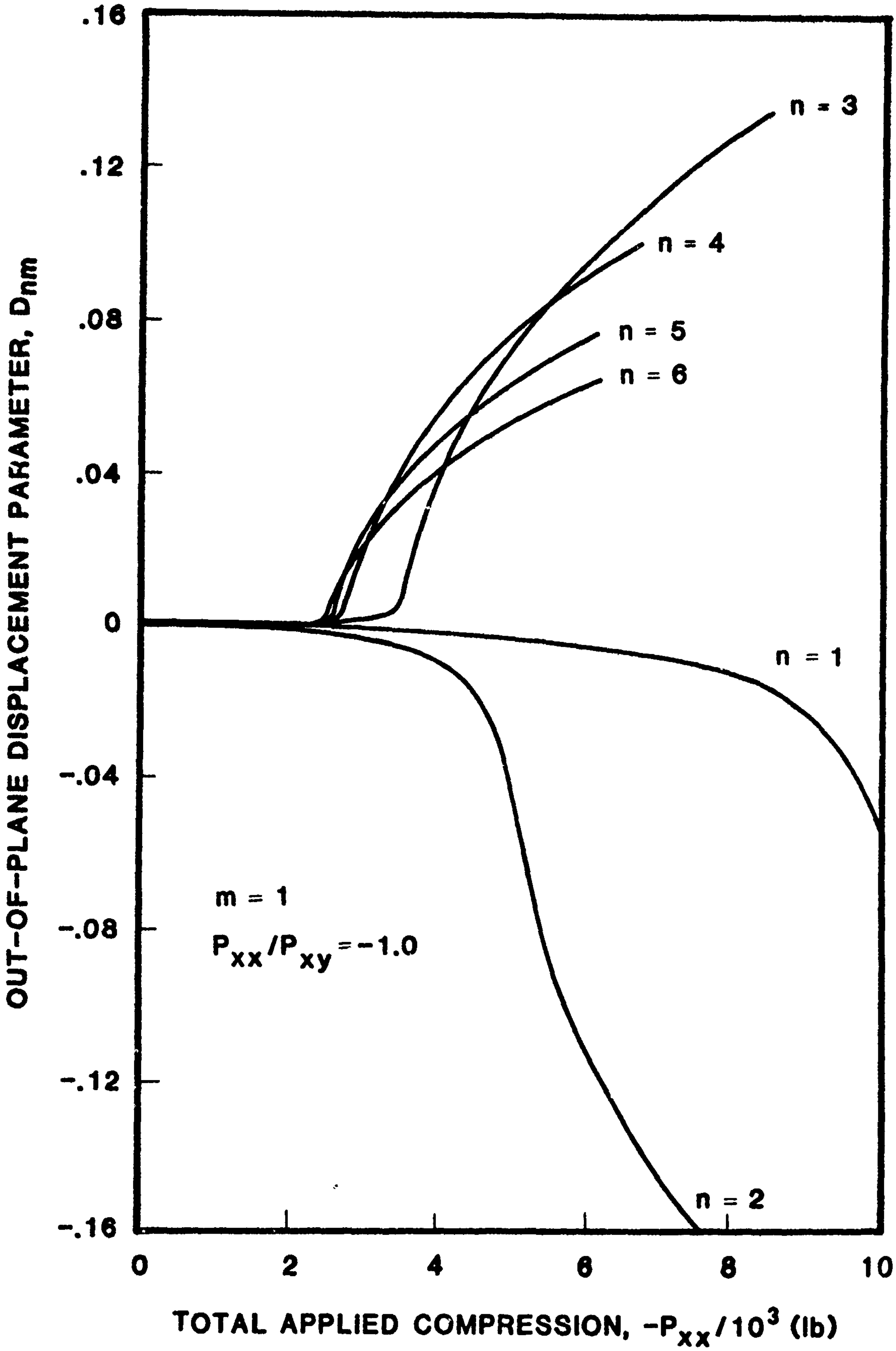


Figure 13. Displacement Coefficient D_{nm} as a Function of the Applied Compression Load.

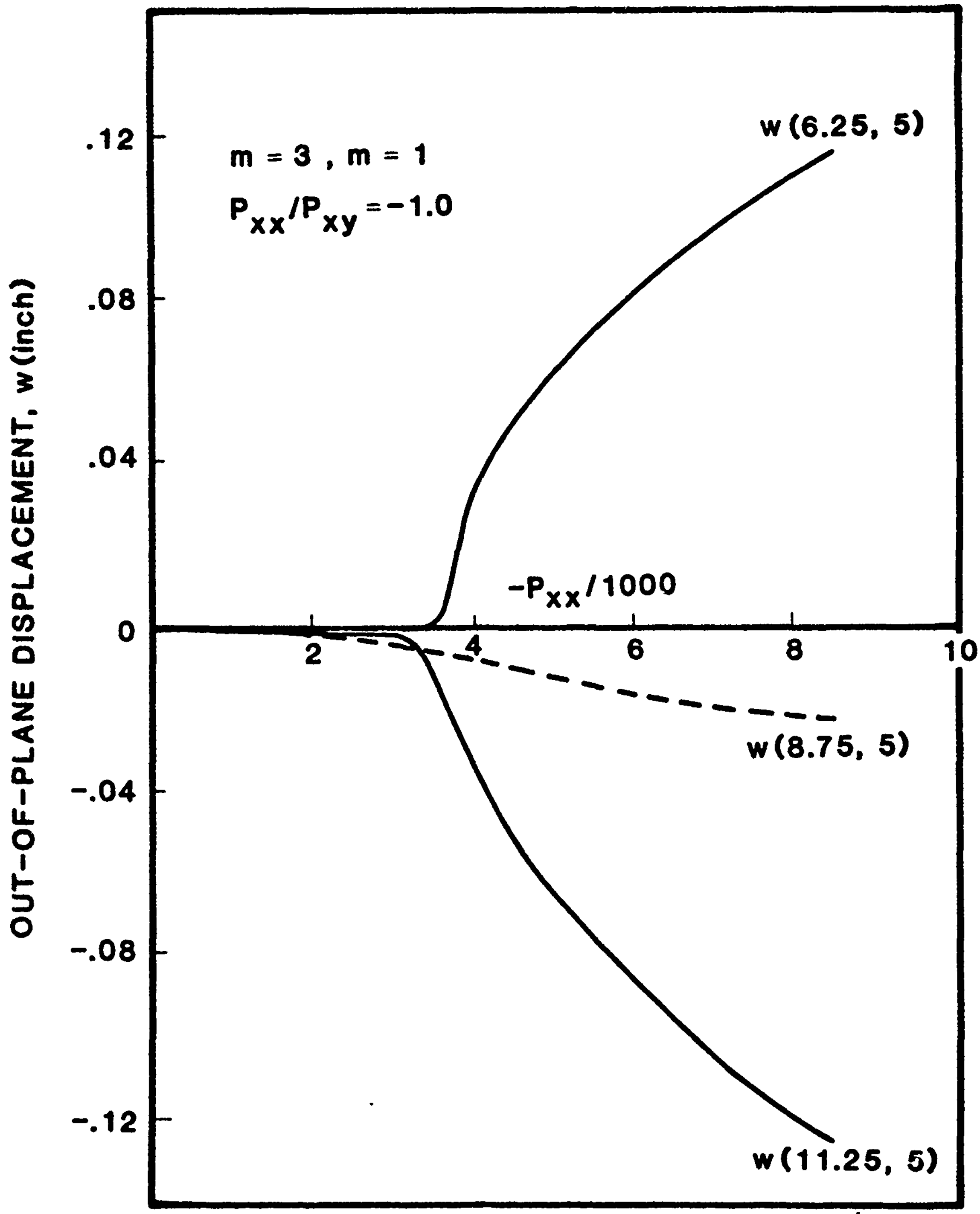


Figure 14. Maximum, Minimum and Panel Center Out-of-Plane Displacements as Functions of the Applied Compression Load.

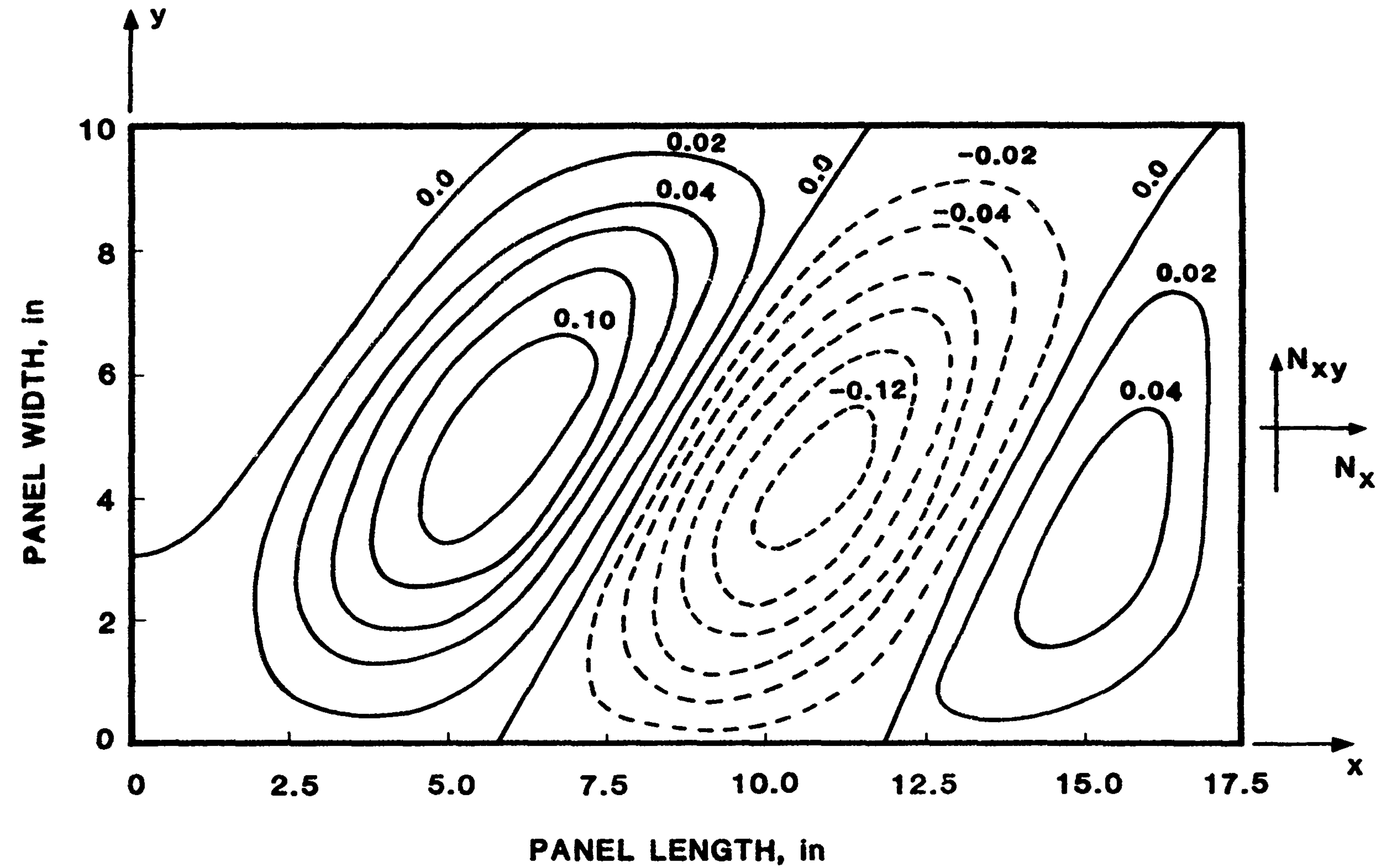


Figure 15. Out-of-Plane Displacement Contours for the Buckling Mode $n = 3$, $m = 1$, at an Applied Compression Load of 8500 lbs. ($N_x/N_{xy} = 1.0$).

varies linearly with the applied compression for $n = 1$. For $n \geq 2$, the b_1 curves are slightly nonlinear. The values of b_1 are approximately equal for $n = 3$ to 6.

The in-plane displacement parameters A_{nm} and b_{nm} are shown in Figures 10 and 11. The coefficient A_{nm} shown in Figure 10 is a measure of the in-plane/out-of-plane displacement interaction in the x-direction. As shown in the figure, for $n \geq 3$, this parameter remains approximately zero when the applied load is below the initial buckling load. At higher loads the absolute value of the parameter increases with applied load. For $n = 1$ and 2, the absolute value of A_{nm} continuously increases with applied load. These results again indicate that the panel analyzed is not likely to deform into the first two modes ($n = 1$, 2). Figure 11 shows a similar behavior for the coefficient B_{nm}, the in-plane/out-of-plane displacement interaction coefficient for the v-displacement.

Figures 12 and 13 show the variation of the out-of-plane displacement coefficients C_{nm} and D_{nm}. The trends for these displacement parameters are similar to that of A_{nm} and B_{nm}. However, they are one order of magnitude higher than Anm and Bnm.

Figure 14 shows the maximum, minimum and panel-center out-of-plane displacement variations with the total applied compression load for the buckling mode $n = 1$, $m = 3$. The panel center is at (8.75, 5). As shown in the figure, the out-of-plane displacement at the panel-center remains relatively small as the load increases. This is because the center is in the vicinity of a nodal line for the assumed buckling mode. The maximum (outward) displacement occurs at location (6.25, 5). Figure 14 shows that the displacement at this point remains at approximately zero below a compression load of 3500 lbs. Above 3500 lbs., the displacement increases rapidly as the load approaches 4000 lbs. Beyond 4000 lbs., the displacement increases with load at a relatively slower rate. The minimum (maximum inward) displacement occurs at the location (11.25, 5). The out-of-plane displacement at this point varies with applied compression load in a manner similar to that of the maximum displacement but in the opposite direction.

The out-of-plane displacement contours for the buckling mode $n = 3$, $m = 1$ at applied compression load of 8500 lbs. are shown in Figure 15. The figure shows that there are two major buckles in the center portion of the panel. These buckles are oriented at an angle of approximately 60° from the x-direction. The buckle near the fixed end (left buckle) deforms outward (positive displacement). The buckle near the loading end (right buckle) deforms inward (negative displacement). In addition to these major buckles, two minor adjacent buckles also develop. These are shown at the left-upper and right-lower corner in Figure 15.

The results shown above indicate that the single-mode analysis developed here can be used to investigate the postbuckling behavior of a panel in several ways. First, a parametric study can be carried out to determine the possible buckling modes of a panel. Once a dominating mode is selected, the single-mode analysis can be used to describe the approximate postbuckling displacement pattern of the panel. This analysis provides the approximate displacement and stress (strain) fields of the panel at different load levels. This information can then be used for a fatigue or failure analysis of the panel.

3.3 MULTI-MODE ANALYSIS

In the single mode analysis, the solution is obtained for a preselected buckling mode. The implicit assumption in the single mode analysis is that the panel deforms in a fixed mode as the applied load increases in the postbuckling regime. This assumption has been adopted in a number of analyses in the literature. However, experimental data indicated that buckling mode shape may change as the applied load increases. In order to simulate the change of buckling mode in the postbuckling regime, a multi-mode analysis method is needed.

The multi-mode analysis method developed is based on the strain energy method discussed in 3.1. However, a general solution of Equation 58 is prohibitive because of the excessive computing resource requirements and convergence problems in the numerical solution. The number of nonlinear

equations in Equation 58 is 4NM+2 for NxM modes used in the general analysis. The number of energy integrals increases rapidly with N and M. Table 2 shows the total number of energy integrals required for different values of N and M. The numbers shown in Table 2 include the initial imperfection terms. These numbers suggest that selected buckling modes up to 4 x 1 would be more practical.

In addition to the number of energy integrals, the number of equations also limits the number of buckling modes used in the analysis. This is because the interaction of buckling modes results in convergence problems in solving the nonlinear equations. In the multi-mode solution, the two numerical procedures discussed in 3.1 for either solving Equations 58 or directly minimizing the total potential energy are used alternately to avoid a convergence problem.

The analysis procedure for the multi-mode solution was coded in a Fortran computer program PACL (Postbuckling Analysis for Combined Loads). Details of the computer program are documented in a separate volume of this report - Automated Data Systems Documentation (Reference 14).

Table 2. Number of Integrals Required.

NUMBER OF N TERMS	NUMBER OF M TERMS	NUMBER OF CONSTANTS
1	1	230
2	1	1222
3	1	3976
4	1	9944
4	2	110,956
5	1	21,034
6	1	39,610
6	2	498,112
7	1	68,492
8	1	110,956
8	4	21,725,092

SECTION 4

ANALYSIS AND TEST RESULTS CORRELATION

4.1 Introduction

The static and fatigue data presented in Reference 15 were analyzed to correlate the measured initial buckling load, ultimate strength and failure mode with predictions from the semi-empirical analysis methodology. The measured strains, buckling mode shape changes and panel stiffness changes were compared with predictions from the energy method based analysis presented in Section 3. The fatigue life data were utilized to establish S-N curves for metal and composite panels. These results are discussed in the following paragraphs.

4.2 Initial Buckling Under Combined Loads

Initial buckling predictions for the metal panels were based on the semi-empirical expressions (Reference 6) given in Equations 1 and 3. A comparison of the buckling load data obtained in this program for pure shear or pure compression loading, with the predictions is shown in Figure 16. As seen in the figure, the semi-empirical predictions are conservative by as much as 30 percent. This observation is consistent with the metal panel data generated in Reference 1.

Under combined loading, the buckling loads were predicted using a parabolic interaction. A comparison of the predictions and the test data is shown in Figure 17. Since the pure shear and pure compression buckling loads were higher than predicted, the buckling loads under combined compression and shear are underestimated by Equation 4. A true comparison, however, can be seen in Figure 18 where the interaction is shown in terms of the buckling load ratios R_c and R_s. The comparison in Figure 18 shows excellent correlation between the predictions and the test data provided the scatter in the measured values for the different panels is accounted for by normalization. Reference 16 data, used to establish the parabolic interaction equation, are also shown to illustrate the consistency of the data obtained in the present program.

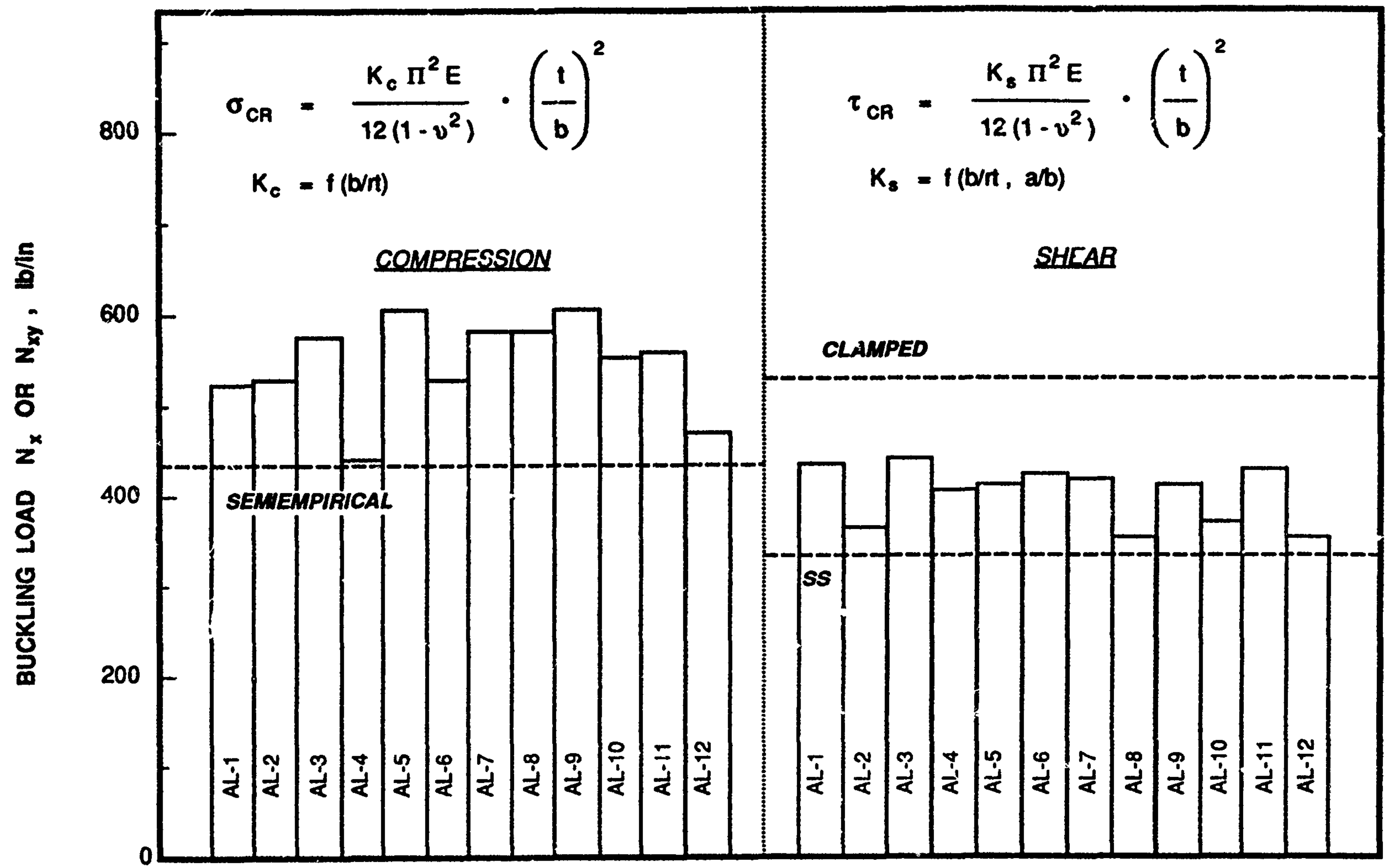


Figure 16. Comparison of Metal Panel Pure Shear and Pure Compression Buckling Loads with Predictions.

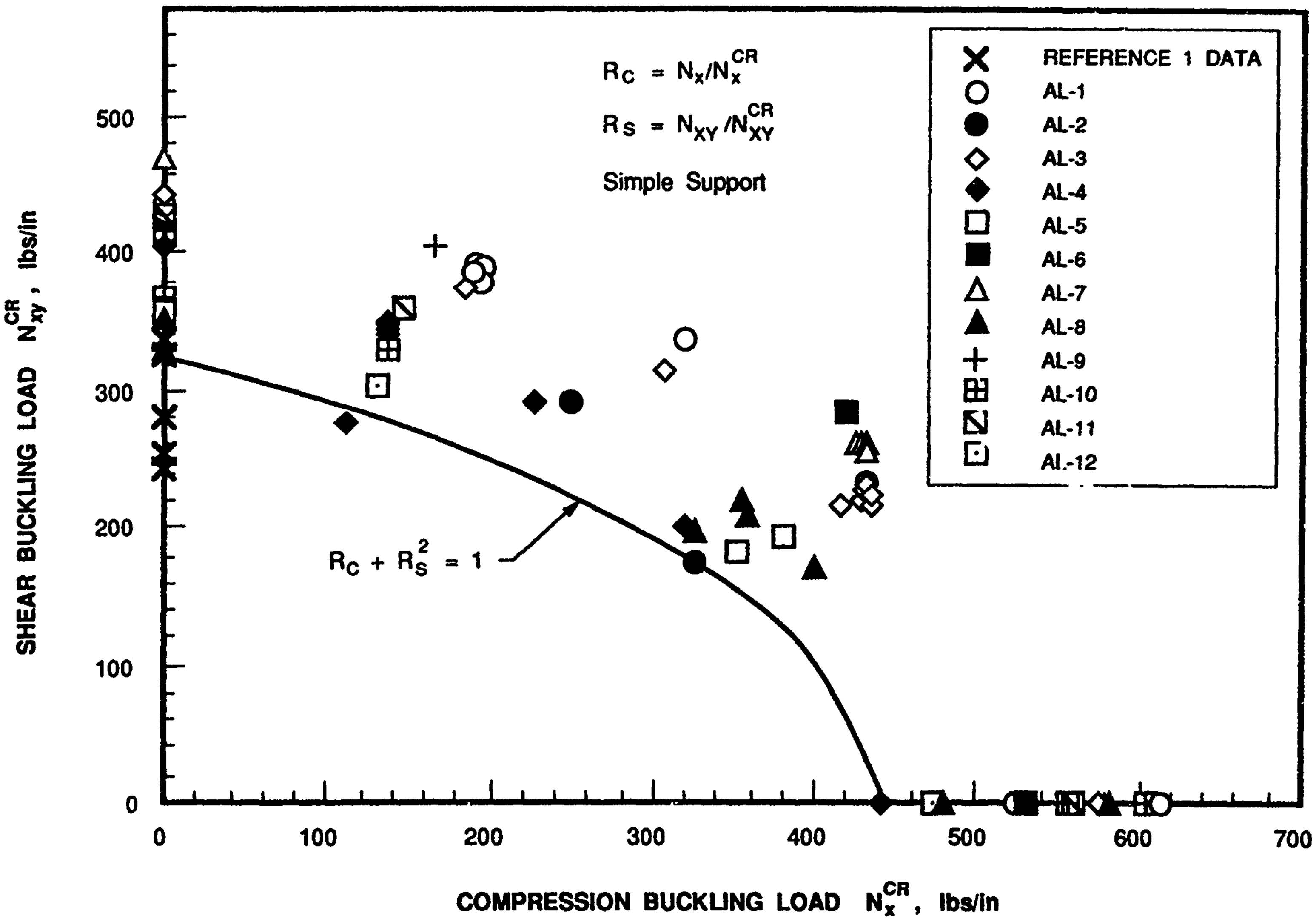


Figure 17. Combined Load Buckling Data for Metal Panels and Comparison with Predictions.

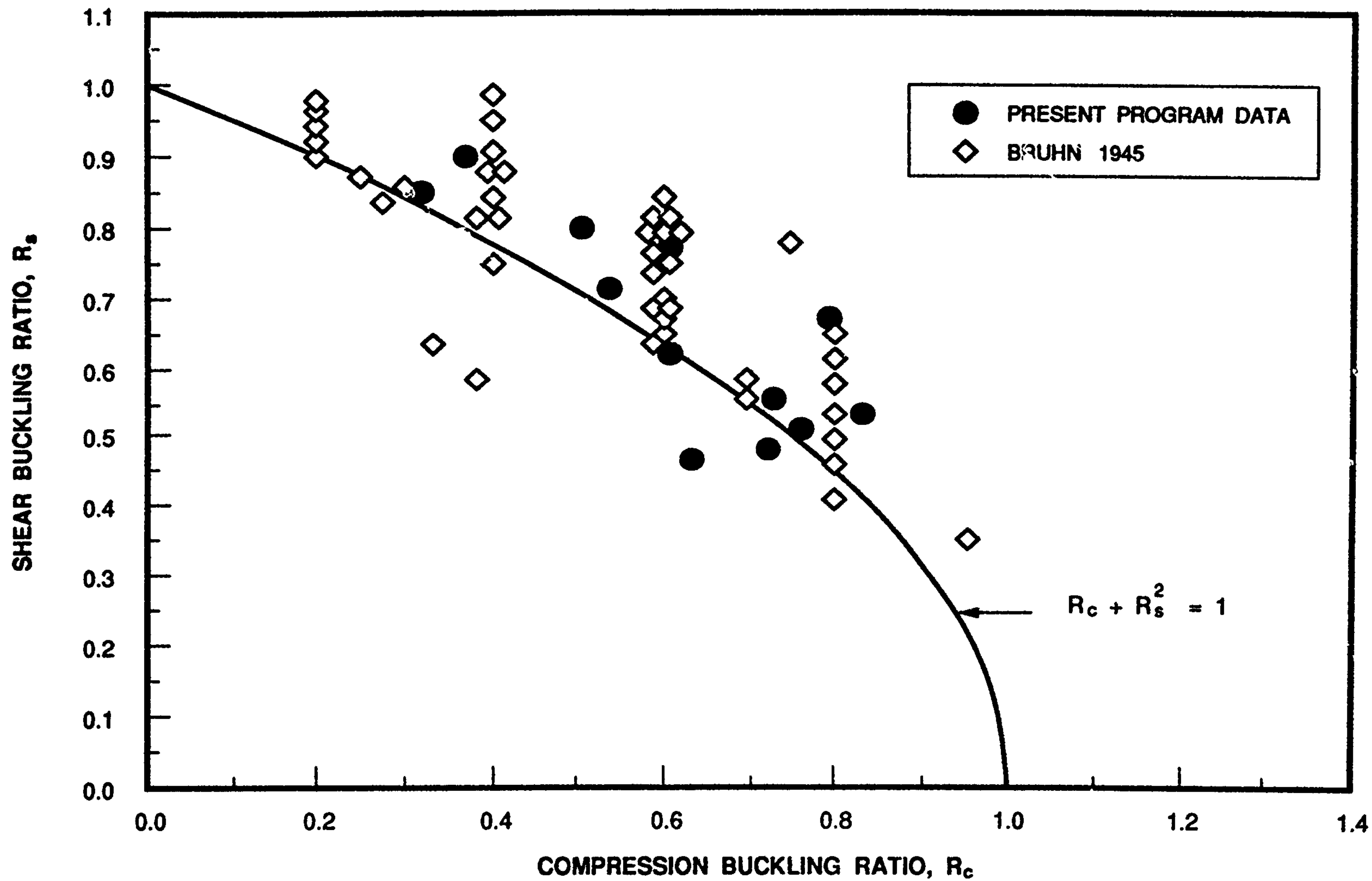


Figure 18. Correlation of Normalized Buckling Loads with Parabolic Interaction Predictions.

Figure 18 shows that the buckling interaction equation used provides a lower bound and is thus somewhat conservative for buckling load predictions for metal panels under combined shear and compression loading.

The correlation between initial buckling load predictions and test data for composite panels is shown in Figure 19. In the case of composite panels, the pure shear, pure compression and the combined loading initial buckling predictions were based on program SS8 (Reference 8). In Figure 19, the parabolic and linear interaction curves are also shown for comparison. The linear interaction expression provides a lower bound for the test data. For preliminary design purposes, use of the linear interaction is more appropriate for composite panels. Figure 20 shows that the test data are bounded by the linear and a fourth power (i.e., $\alpha = 1$ and $\alpha = 4$ in the expression $R_c + R_s^{\alpha} = 1$).interaction rule (Reference 17).

4.3 ULTIMATE STRENGTH UNDER COMBINED LOADS

The ultimate strength of metal and composite panels was predicted using the methodology given in Section 2. The strength predictions were plotted as failure envelopes and are shown in Figures 21 and 22 for metal and composite panels, respectively. The only change in the strength prediction methodology made after comparison with test data was in the stiffener crippling interaction equation under combined loading. Originally (Equation 41) the following criterion was adopted for stiffener crippling:

$$\frac{\varepsilon_s^{co}}{\varepsilon_s^{cc}} + \left(\frac{\varepsilon_s^{so}}{\varepsilon_{os}} \right)^{1.5} \leq 1.0$$

However, the test data for both metal and composite panels show better correlation with a linear interaction, i.e.,

$$\frac{\varepsilon^{co}}{\varepsilon_s^{cc}} + \left(\frac{\varepsilon_s^{so}}{\varepsilon_{os}} \right) \leq 1 \tag{61}$$

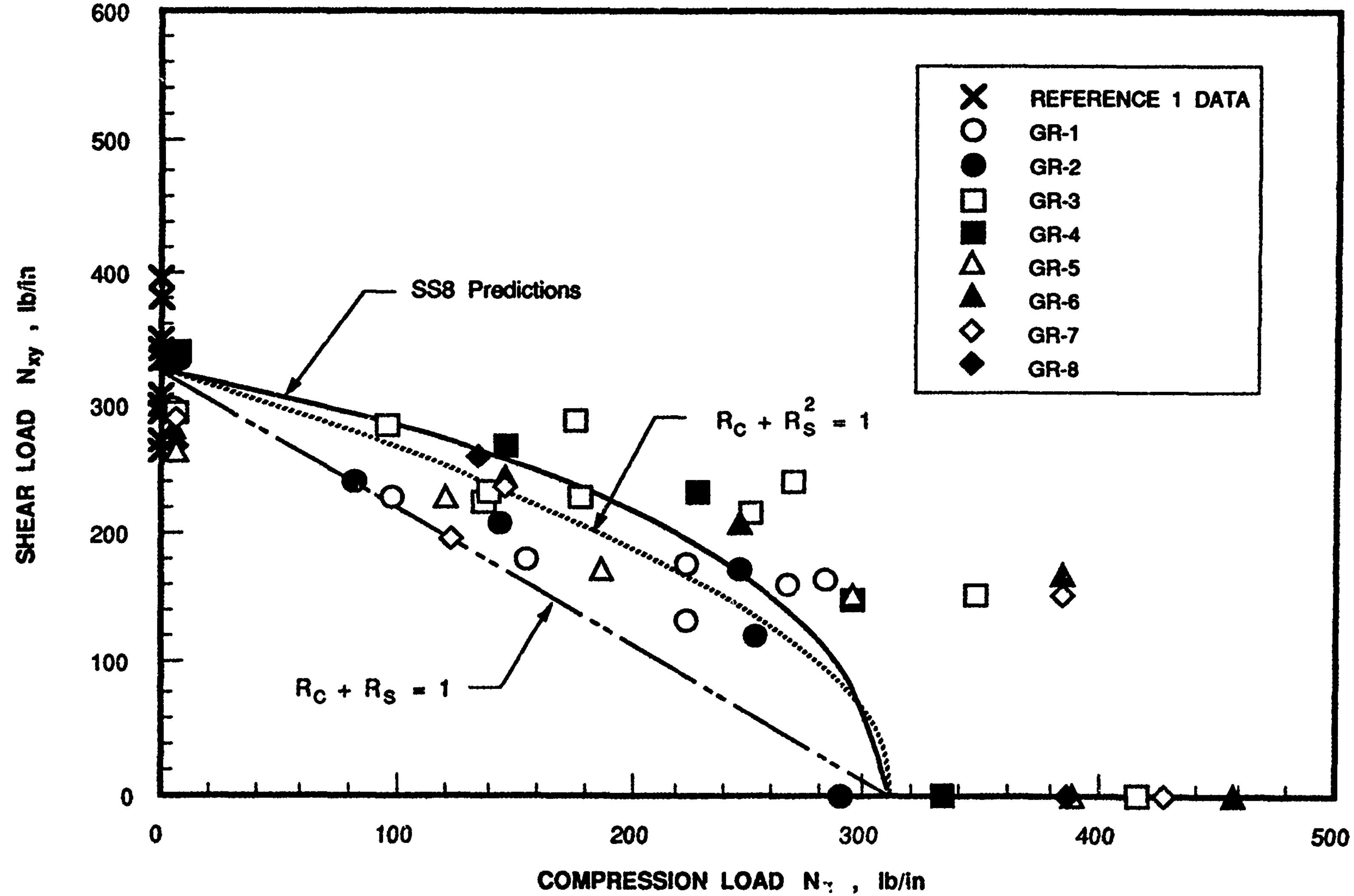


Figure 19. Comparison of Measured and Predicted Buckling Loads for Composite Panels.

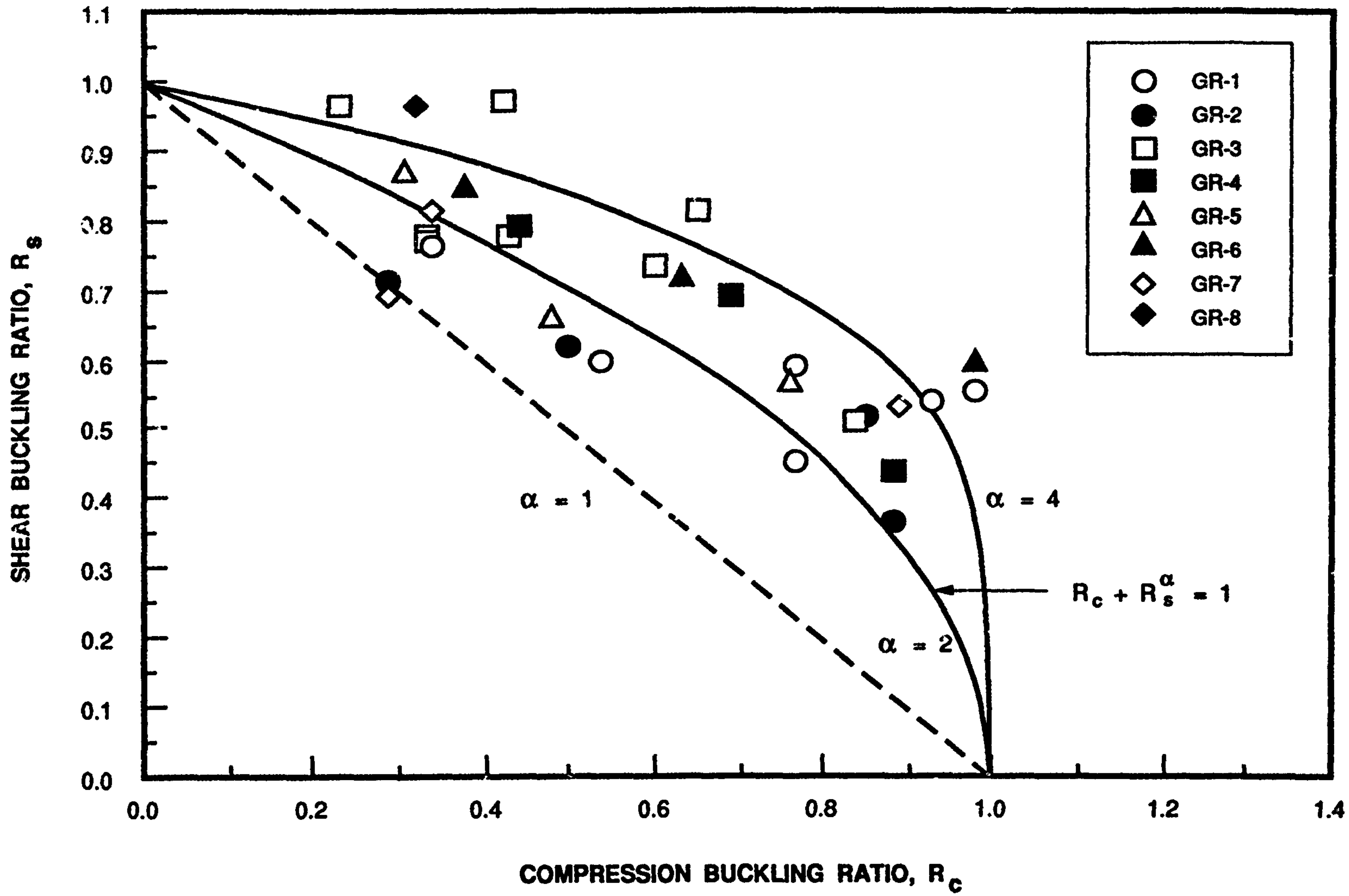


Figure 20. Comparison of Measured Buckling Loads with Predictions from Different Interaction Rules.

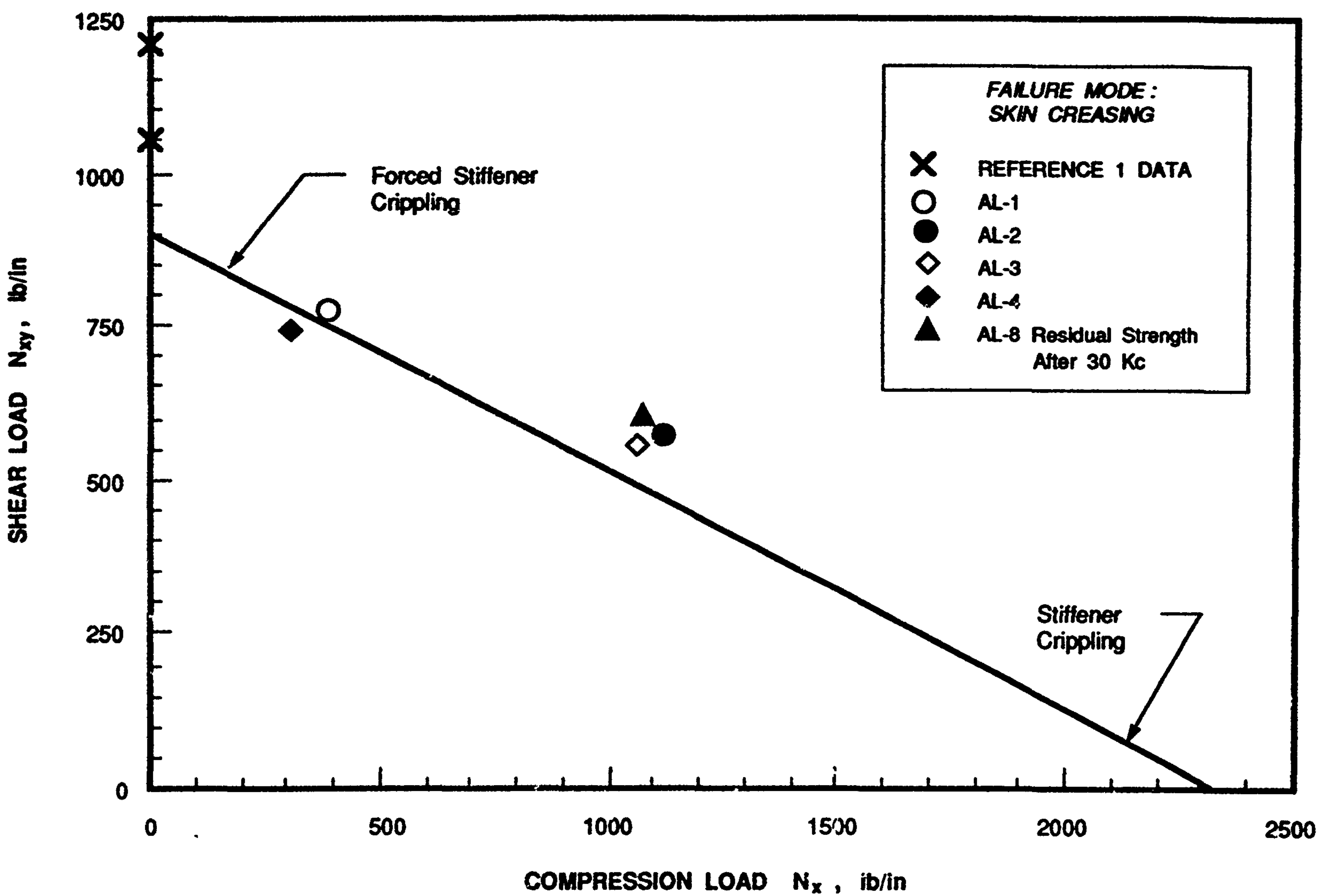


Figure 21. Failure Envelope for Metal Panels and correlation with Test Data.

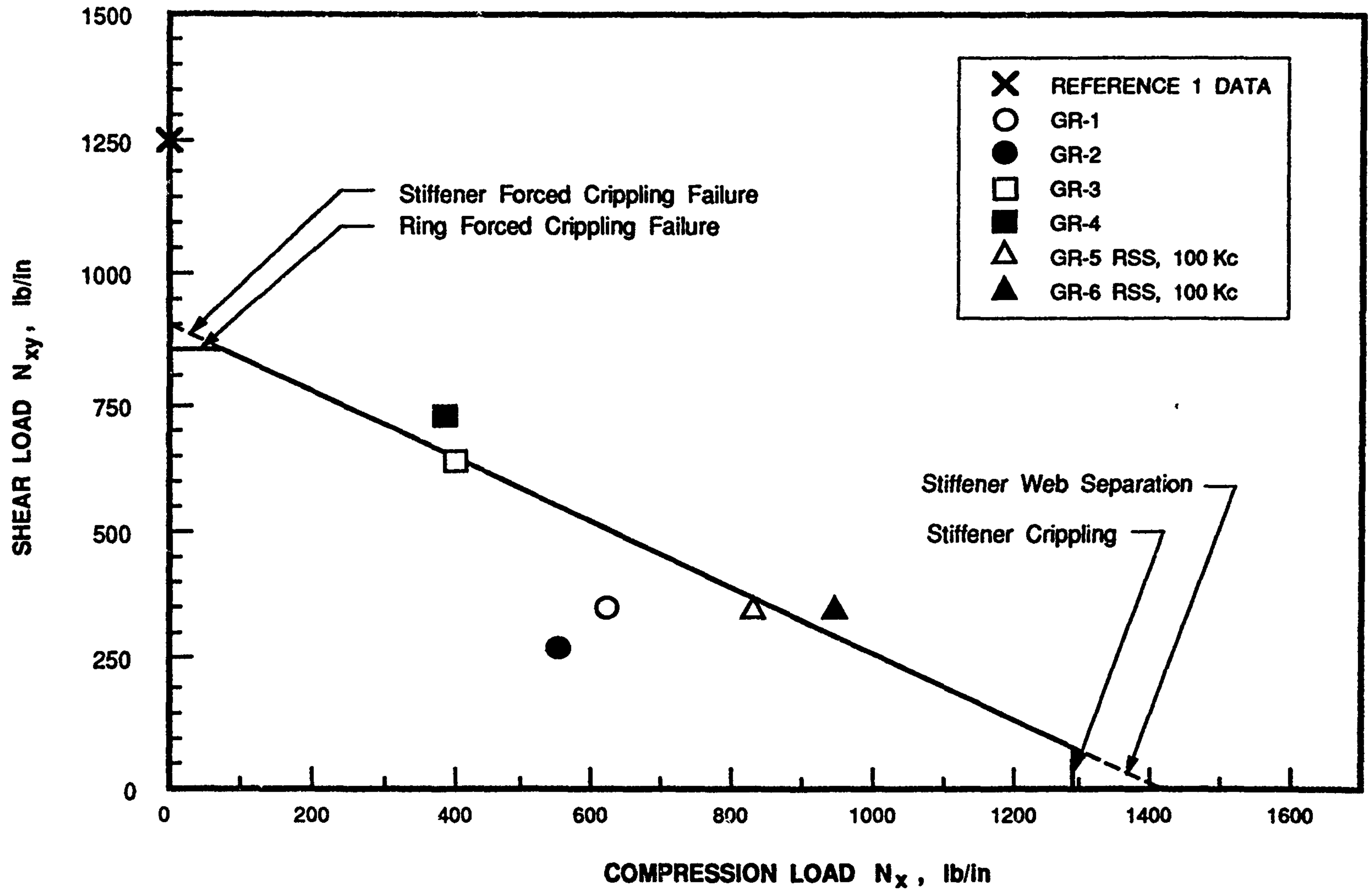


Figure 22. Failure Envelope for Composite Panels and Correlation with Test Data.

As shown in Figure 21 the metal panel strength data show excellent correlation with predictions. The failure mode, however, was permanent set in the skin. The close agreement of the test data with stiffener crippling predictions leads to the conclusion that the skin creasing was precipitated by stiffener crippling.

The composite panel test data also show good agreement with the linear interaction stiffener crippling prediction. There are two exceptions, however, in panels GR-1 and GR-2. The low failure loads obtained for these panels are plausible since these two early panels showed some load introduction problems during the static tests. Specifically, the panel load introduction area skin thickness was the same as the test section skin thickness. Due to load introduction eccentricities, the skin in the load introduction area buckled before the panel ultimate load was reached. Thus, the two panels were not subjected to a uniform axial compression load and, therefore, showed failure loads slightly lower than the predictions. In all other panels the load introduction region thickness was increased by secondarily bonding fiberglass laminates. Thus, the semi-empirical design method as given in Section 2 with Equation 61 replacing Equation 41 can be used for designing curved composite panels under uniaxial compression and shear loads.

4.4 FATIGUE LIFE UNDER COMBINED LOADS

The fatigue test data for metal and composite panels are fully documented in Reference 15. A summary of the fatigue failure modes for the metal panels under compression dominated constant amplitude loading i.e. $(N_x)_{max}/(N_{xy})_{max} = 2$ with R=10 for compression and R=-1 for shear*, is shown in Table 3. For shear dominated constant amplitude fatigue loading i.e. $(N_x)_{max}/(N_{xy})_{max} = 0.5$ with R=10 for compression and R=-1 for shear, the test results and failure modes are shown in Table 4. The basic fatigue failure mode in the metal panels under compression dominated loading was crack initiation in the skin adjacent to the stiffener flange and subsequent propagation along the loading direction. The crack initially propagated along

*Note that due to differences in R-ratios, $(N_x)_{max}$ and $(N_y)_{max}$ do not occur simultaneously.

Table 3. Fatigue Failure Modes of Aluminum Panels Under Compression Dominated Loading.

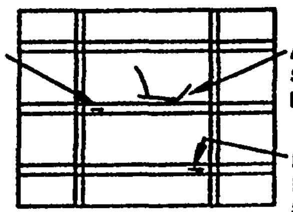
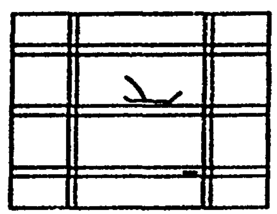
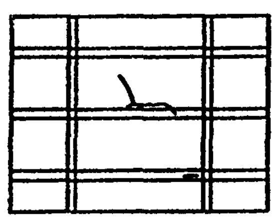
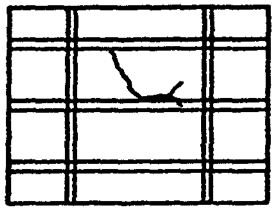
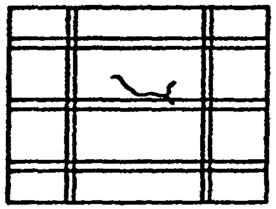
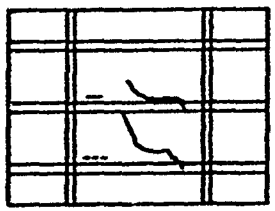
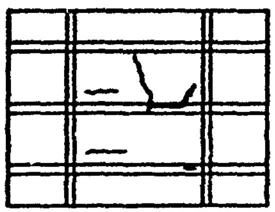
PANEL No.	FATIGUE CRACKS	MAX FATIGUE LOAD, lbs/in N_x	MAX FATIGUE LOAD, lbs/in N_{xy}	$\frac{N_x}{N_{xy}}$	$\frac{N_x}{N_x^{CR}}$	$\frac{N_{xy}}{N_{xy}^{CR}}$	MAX. FATIGUE LOAD, % STATIC STRENGTH	FATIGUE HISTORY	STATIC OR FATIGUE FAILURE
AL-5		640	375	1.70	1.70	1.92	59	CRACK INITIATION AT 24,530 CYCLES	FATIGUE TEST STOPPED, AFTER 41,740 CYCLES WITH CRACK PATTERN SHOWN
AL-6		686	376	1.83	1.64	1.33	63	CRACK INITIATION AT 22,500 CYCLES	FATIGUE TEST STOPPED AFTER 38,905 CYCLES WITH CRACK PATTERN SHOWN
AL-7		605	304	1.99	1.42	1.17	56	RUN OUT AT 100K CYCLES. C.ACKS INITIATED AFTER FATIGUE LOAD WAS INCREASED	FATIGUE FAILURE AFTER 162K CYCLES AT HIGHER LOADS
AL-8		627	362	1.73	1.77	1.65	58	CRACK INITIATION AT 16,604 CYCLES	FATIGUE TEST STOPPED AT 27,900 CYCLES WITH CRACK PATTERN SHOWN

Table 4. Fatigue Failure Modes of Aluminum Panels Under Shear Dominated Loading.

PANEL No.	FATIGUE CRACKS	MAX FATIGUE LOAD, lbs/in N_x	N_{xy}	$\frac{N_x}{N_{xy}}$	$\frac{N_x}{N_x^{CR}}$	$\frac{N_{xy}}{N_{xy}^{CR}}$	MAX. FATIGUE LOAD, % STATIC STRENGTH	FATIGUE HISTORY	STATIC OR FATIGUE FAILURE
AL-9		302	745	0.41	1.82	1.83	0.86	CRACK INITIATION AT 10,635 CYCLES	FATIGUE TEST STOPPED, AFTER 12,273 CYCLES WITH CRACK PATTERN SHOWN
AL-10		225	542	0.41	1.63	1.64	0.64	CRACK INITIATION AT 22,300 CYCLES	FATIGUE TEST STOPPED AFTER 42,971 CYCLES WITH CRACK PATTERN SHOWN
AL-11		253	608	0.42	1.72	1.69	0.73	CRACK INITIATION AT 13,540 CYCLES	FATIGUE TEST STOPPED AFTER 25,120 CYCLES WITH CRACK PATTERN SHOWN
AL-12		280	648	0.43	2.15	2.13	0.80	CRACK INITIATION AT 16,543 CYCLES	FATIGUE TEST STOPPED AT 24,100 CYCLES WITH PATTERN SHOWN

the stiffener direction. After a certain length, the crack branched and grew toward the centerline of the bay in the diagonal direction. The crack initiation life of the panels was approximately 60 percent of the total number of cycles required to tear one skin bay.

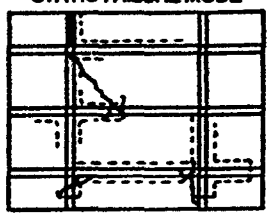
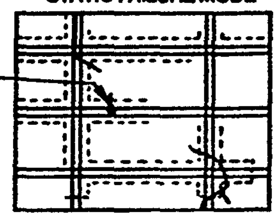
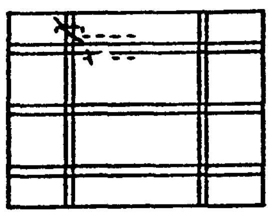
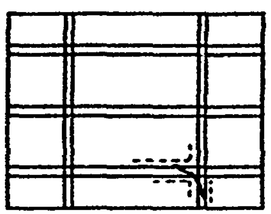
Under shear dominated loading, cracks initiated at the edges of fastener holes in the skin. The subsequent crack growth pattern and the crack initiation life relative to the total number of cycles required to tear one bay were similar to those obtained in the compression dominated tests.

In practical aircraft structures curved panels are most commonly used in pressurized fuselage structures. Therefore, the crack initiation life was defined as the fatigue life of the metal panels. It should be noted, however, from the test data that postbuckled metal panels retain a significant percentage of their static strength even after the loss of skin due to cracking under diagonal tension stresses.

The measured crack initiation life of the metal panels was used to generate the S-N curve shown in Figure 23. In this figure, the typical fatigue failure mode is also illustrated. The limited data generated in this program indicate steep S-N curves for metal panels operating in the postbuckling range. Secondly, the metal panels can sustain static loads of approximately 2.5 times the average buckling load but in actual structures their capability would be limited to 1.25 times the buckling loads due to fatigue considerations.

Fatigue test results for the composite panels are summarized in Table 5. The two panels tested at $(N_x)_{max}/(N_{xy})_{max} = 2$ experienced no fatigue failure after 100,000 cycles of constant amplitude loading. Residual static strength tests on these panels indicated no strength reduction (See Figure 22). The static failure mode was primarily skin/stiffener separation. Panels under constant amplitude shear dominated loads i.e. $(N_x)_{max}/(N_{xy})_{max} = 0.5$, failed under fatigue cycling. The fatigue failure mode in these panels (GR-7 and GR-8) was skin stiffener separation at stiffener and ring intersection accompanied by local skin rupture, Thus, the composite panels appear to be more sensitive in fatigue to shear dominated loading.

Table 5. Fatigue Failure Modes for Composite Panels.

PANEL No.	FATIGUE CRACKS*	MAX FATIGUE LOAD, lbs/in N_x	MAX FATIGUE LOAD, lbs/in N_{xy}	$\frac{N_x}{N_{xy}}$	$\frac{N_x}{N_x^{CR}}$	$\frac{N_{xy}}{N_{xy}^{CR}}$	MAX. FATIGUE LOAD, % STATIC STRENGTH	FATIGUE HISTORY	STATIC OR FATIGUE FAILURE
GR-5	STATIC FAILURE MODE 	538	218	2.47	1.82	1.45	0.69	RUN OUT AT 100,000 CYCLES	RESIDUAL STATIC STRENGTH: N_x = 833 lb/in N_{xy} = 353 lb/in
GR-6	STATIC FAILURE MODE STIFF. FAILURE 	590	239	2.70	1.68	1.41	0.76	RUN OUT AT 100,000 CYCLES	RESIDUAL STATIC STRENGTH: N_x = 810 lb/in N_{xy} = 356 lb/in
GR-7		334	530	0.63	2.29	2.26	0.78	FATIGUE FAILURE AS SHOWN AT 38,743 CYCLES	FATIGUE
GR-8		286	521	0.55	2.13	2.03	0.77	FATIGUE FAILURE AS SHOWN AT 62,095 CYCLES	FATIGUE

* —— SKIN RUPTURE ------- STIFFENER/SKIN DISBONDS

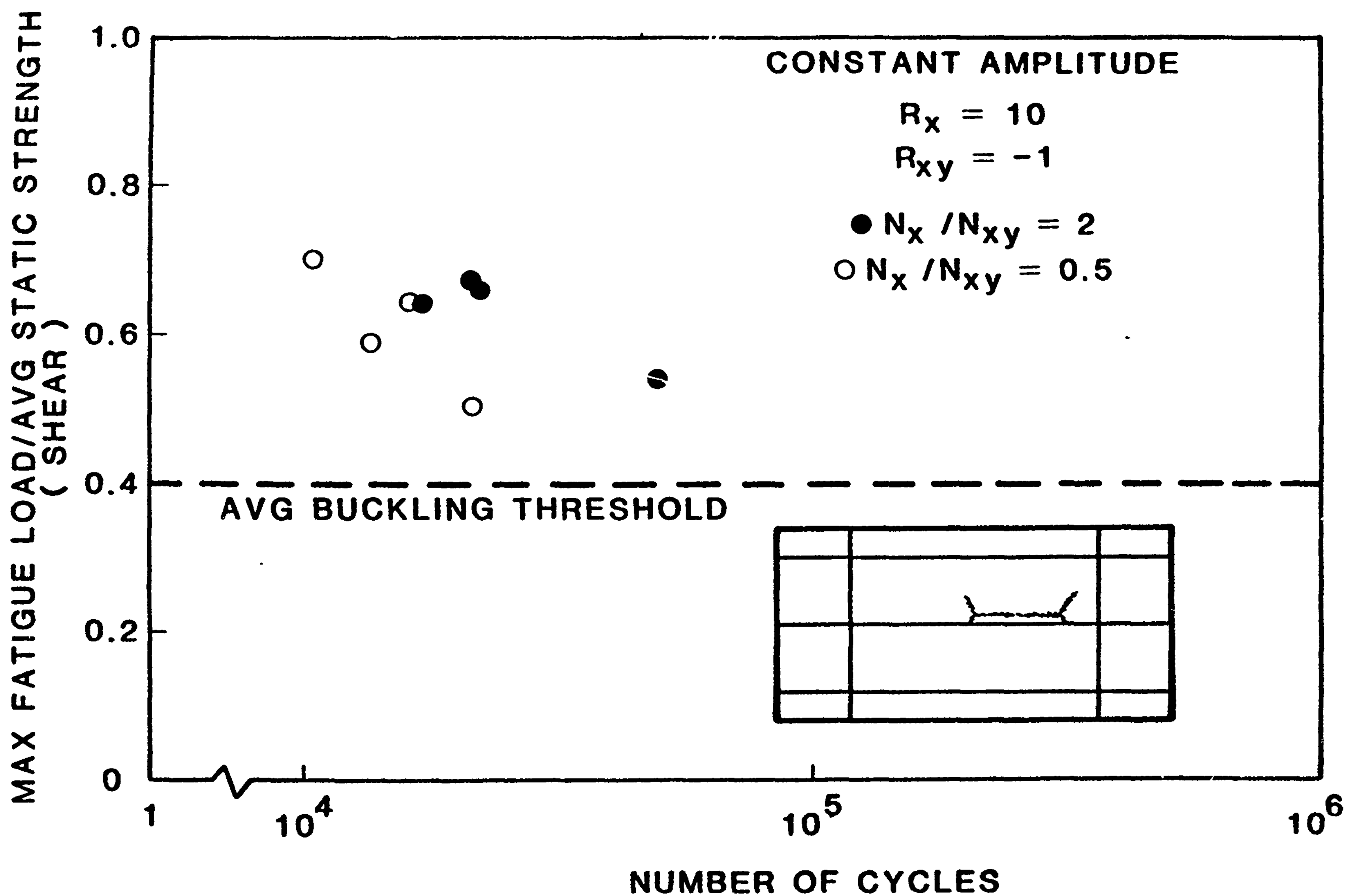


Figure 23. Metal Panel Fatigue Data.

Figure 24 shows a plot of the number of fatigue cycles sustained by the composite panels versus the applied loads. From four data points in Figure 24 a fatigue threshold was estimated to be approximately 80 percent of the static strength. The fatigue advantage of composite panels relative to metal panels is readily apparent from Figure 24 in that the composite panels could be utilized up to 200 percent of their initial buckling load for shear dominated loading as opposed to 125 percent for metal panels. The postbuckling range for composite panels under compression dominated loading could be possibly higher.

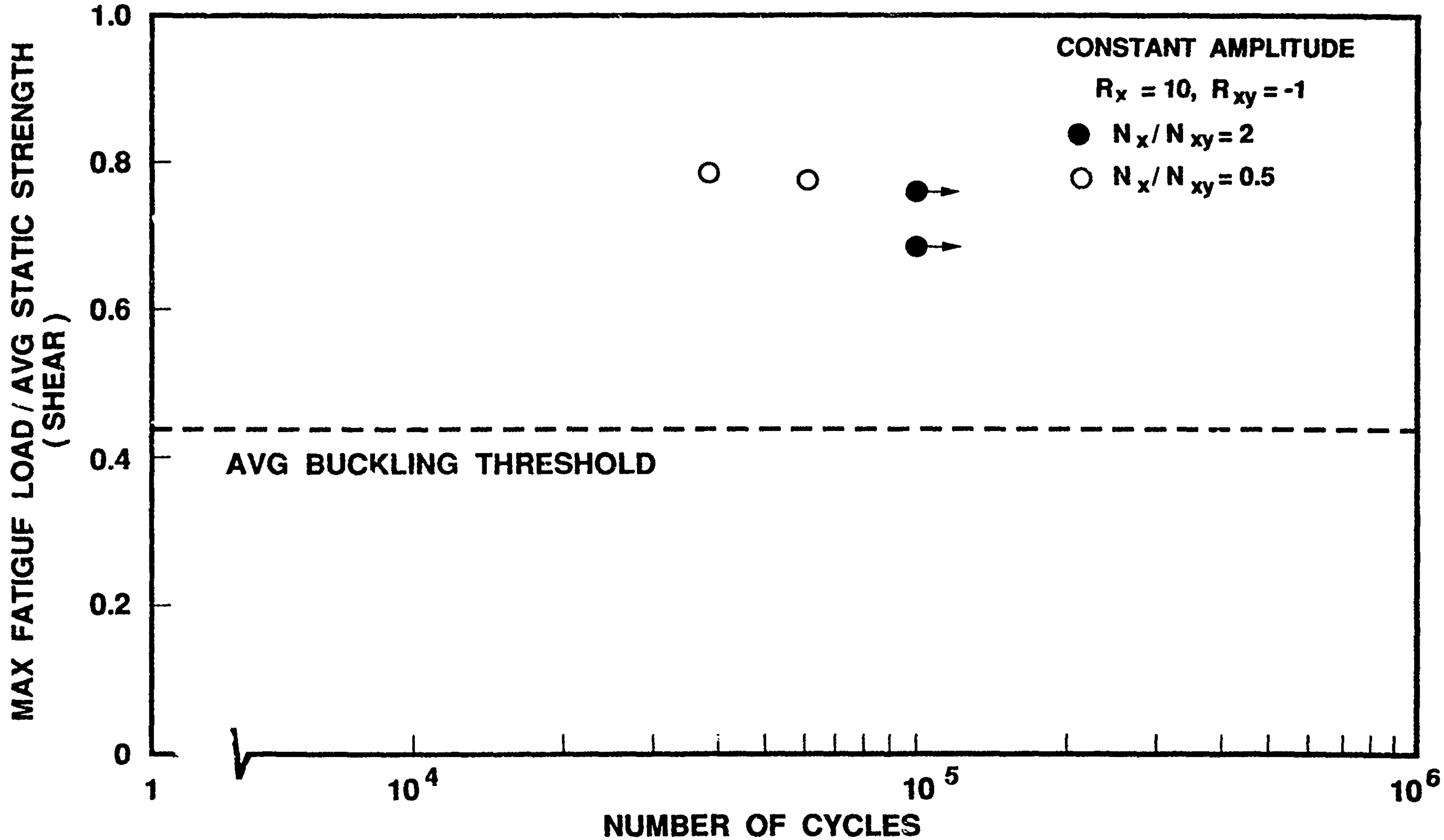


Figure 24. Composite Panel Fatigue Data.

SECTION 5

CONCLUSIONS

The significant conclusions from this program are summarized in the following paragraphs.

5.1 SEMI-EMPIRICAL DESIGN METHODOLOGY FOR POSTBUCKLED PANELS UNDER COMBINED LOADS

1. The semi-empirical static design methodology developed in Reference 1 for postbuckled composite and metal panels under pure shear or pure compression loading was extended to panels under combined uniaxial compression and shear loads.
2. The methodology was coded in a computer program (PBUKL) for rapid iterative design of composite and metal panels.
3. Experimental verification data were used to develop a new criteria to predict the effect of shear and compression load interaction on composite panel skin buckling. A linear interaction, although conservative, seems more appropriate for the design of composite panels as opposed to the well established parabolic interaction rule for metal panels.
4. The test data showed that for both composite and metal panels a linear interaction rule for stiffener crippling prediction yields better correlation than a non-linear interaction rule.
5. Ultimate panel strength predictions based on the semi-empirical analysis for composite and metal panels were found to be very accurate and well suited for design purposes.
6. Stiffener and skin separation in composite panels was the observed failure mode under static combined uniaxial compression and shear loading.
7. For metal panels under combined compression and shear loading stiffener crippling was the dominant failure mode. Permanent deformation of the skin was either concurrent or precipitated by stiffener crippling.
8. Under constant amplitude fatigue loading metal panel failure occurred by crack initiation in the skin adjacent to a

stiffener. The static strength of the metal panels was unaffected by the skin crack propagating across an entire bay. The crack initiation life was approximately 60 percent of total number of cycles sustained by the panel prior to skin rupture in a single bay. The nature of the combined loading, i.e. compression dominated or shear dominated, did not affect the crack propagation pattern. However, the initiation sites in the two cases were different. Under shear dominated loading the cracks initiated at stiffener attachment fastener holes, whereas under compression dominated loading the cracks initiated in the skin at the edges of the stiffener flange attached to the skin.

9. Durability considerations can severely limit the postbuckled operation range of metal panels. In the panel design tested the static strength range was 250 percent of the average initial buckling load. However, for a 100,000 cycle constant amplitude fatigue life, the panel loads would have to be restricted to 125 percent of the initial buckling load.
10. Composite panels demonstrated a high fatigue threshold relative to the initial skin buckling loads. Composite panels designed for a static strength equal to 250 percent of the initial skin buckling load can be safely operated under fatigue loading up to 200 percent of the initial buckling load.
11. Composite panels tested in the program showed a greater sensitivity to shear dominated fatigue loading as compared with compression dominated fatigue loading.
12. The fatigue failure mode in composite panels was separation between the cocured stiffener and the skin. In particular, the region at the intersection of the stiffener and the ring was vulnerable to the failure mode.
13. Repeated buckling had no influence on initial skin buckling loads for either the composite or the metal panels.
14. The semi-empirical design methodology was used to develop a design procedure for composite and metal panels under combined uniaxial compression and shear loading.

5.2 NON-EMPIRICAL ANALYSIS OF POSTBUCKLED PANELS UNDER COMBINED LOADING

1. A single-mode and multi-mode energy method based postbuckling analysis was developed.

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APPENDIX A
ANALYSIS DETAILS

This Appendix defines the individual energy integrals used in Equations (54) through (57), details the nonlinear system given by equation (58) and presents closed form expressions of all the energy integrals.

The displacement functions are expressed in a general form in Equation (53). The following notations are used for the derivatives of the displacement functions:

$$
\begin{aligned}
f_{i,0}\ (\xi,\eta;n,m) &= f_1\ (\xi,\eta;n,m) \\
f_{i,1}\ (\xi,\eta;n,m) &= f_{1,\xi}\ (\xi,\eta;n,m) \\
f_{i,2}\ (\xi,\eta;n,m) &= f_{1,\eta}\ (\xi,\eta;n,m) \\
f_{i,3}\ (\xi,\eta;n,m) &= f_{1,\xi\xi}\ (\xi,\eta;n,m) \\
f_{i,4}\ (\xi,\eta;n,m) &= f_{1,\eta\eta}\ (\xi,\eta;n,m) \\
f_{i,5}\ (\xi,\eta;n,m) &= f_{1,\xi\eta}\ (\xi,\eta;n,m)
\end{aligned}
\qquad \text{(A-1)}
$$

$$i = 1,\ 2,\ 3,\ 4$$

$$f_{5,\alpha}\ (\xi,\eta;n,m) = f_{5,\alpha}\ (\xi,\eta)$$

$$\alpha = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

The individual integrals F, G, H and I in equation (54) are defined as

$$\overset{0}{F}_{i,nm} = \int_0^1\int_0^1 f_i\ (\xi,\eta;n,m)d\xi d\eta \qquad \text{(A-2)}$$

$$i = 1,\ 2,\ 3,\ 4,\ 5$$

$$\overset{\alpha}{F}_{i,nm} = \int_0^1\int_0^1 f_{i,\alpha}\ (\xi,\eta;n,m)d\xi d\eta \qquad \text{(A-3)}$$

$$i = 1,\ 2,\ 3,\ 4,\ 5$$

$$\alpha = 1,\ 2,\ 3,\ 4,\ 5$$

$$G^{\alpha\beta}_{ij,nmpq} = \int_0^1 \int_0^1 f_{i,\alpha}(\xi,\eta;n,m) f_{j,\beta}(\xi,\eta;p,q) d\xi d\eta$$

$$i,j = 1,\ 2,\ 3,\ 4,\ 5 \qquad \text{(A-4)}$$

$$\alpha,\beta = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

$$H^{\alpha\beta\gamma}_{ijk,nmpqrs} = \int_0^1 \int_0^1 f_{i,\alpha}(\xi,\eta;n,m) f_{j,\beta}(\xi,\eta;p,q)$$

$$x f_{k,\gamma}(\xi,\eta;r,s) d\xi d\eta \qquad \text{(A-5)}$$

$$i,j,k = 1,\ 2,\ 3,\ 4,\ 5$$

$$\alpha,\beta,\gamma = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

$$I^{\alpha\beta\gamma\delta}_{ijk\iota,nmpqrstu} = \int_0^1 \int_0^1 f_{i,\alpha}(\xi,\eta;n,m) f_{j,\beta}(\xi,\eta;p,q) \qquad \text{(A-6)}$$

$$x f_{k,\gamma}(\xi,\eta;r,s) f_{\iota,\delta}(\xi,\eta;t,u) d\xi d\eta$$

$$i,j,k,\iota = 1,\ 2,\ 3,\ 4,\ 5$$

$$\alpha,\beta,\gamma,\delta = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

The integrals J and K appeared in Equations (55) - (57) are defined as

$$J^{1}_{1nm} = \int_0^1 f_{1,\xi}(\xi,0;n,m) d\xi \qquad \text{(A-7)}$$

$$J_{1nm} = \int_0^1 f_1(1,\eta;n,m) d\eta$$

$$J_{2nm} = \int_0^1 f_2(1,\eta;n,m) d\eta$$

$$K^{11}_{11nmpq} = \int_0^1 f_{1,\xi}(\xi,0;n,m) f_{1,\xi}(\xi,0;p,q) d\xi \qquad \text{(A-8)}$$

$$K^{33}_{22nmpq} = \int_0^1 f_{2,\xi\xi}(\xi,0;n,m) f_{2,\xi\xi}(\xi,0;p,q) d\xi$$

$$K^{22}_{22nmpq} = \int_0^1 f_{2,\eta}(1,\eta;n,m) f_{2,\eta}(1,\eta;p,q) d\eta$$

$$K^{44}_{11nmpq} = \int_0^1 f_{1,\eta\eta}(1,\eta;n,m) f_{1,\eta\eta}(1,\eta;p,q) d\eta$$

The system of nonlinear algebraic Equations (58) are obtained by setting the derivative of the total potential energy with respect to each of the unknown coefficient to zero. Six groups of equations are obtained and they are given below. In the following equations the total load, P_{xx} and P_{xy} are used in place of bN_{xx} and bN_{xy}.

$$\frac{\partial \Pi}{\partial A_{ij}} = 0$$

$$A_{nm} \left[\frac{b}{a} A^*_{11} G^{11}_{11ijnm} + \frac{a}{b} A^*_{66} G_{11ijnm} + \frac{A_s E_s}{a} K^{11}_{11ijnm} + \frac{I_f E_f}{b^3} K^{44}_{11ijnm} \right]$$

$$+ B_{nm} \left[A^*_{12} G^{12}_{12ijnm} + A^*_{66} G^{21}_{12ijnm} \right]$$

$$+ C_{nm} \left[\frac{bw_o}{a^2} A^*_{11} H^{111}_{135ijnm} + \frac{b}{R} A^*_{12} G^{10}_{13ijnm} + \frac{w_o}{b} A^*_{12} H^{122}_{135ijnm} \right.$$

$$\left. + \frac{w_o}{b} A^*_{66} (H^{221}_{135ijnm} + H^{212}_{135ijnm}) \right]$$

$$+ D_{nm} \left[\frac{bw_o}{a^2} \overset{*}{A}_{11} H^{111}_{145ijnm} + \frac{b}{R} \overset{*}{A}_{12} G^{10}_{14ijnm} + \frac{w_o}{b} \overset{*}{A}_{12} H^{122}_{145ijnm} \right.$$

$$\left. + \frac{w_o}{b} \overset{*}{A}_{66} (H^{212}_{145ijnm} + H^{221}_{145ijnm}) \right]$$

$$+ a_1 \left[b \overset{*}{A}_{11} F^{1}_{1ij} + A_s E_s J^{1}_{1ij} \right] + b_1 (a \overset{*}{A}_{66} F^{2}_{1ij})$$

$$= P_{xx} J_{1ij} - C_{nm} C_{pq} \left[\frac{b}{2a^2} \overset{*}{A}_{11} H^{111}_{133ijnmpq} + \frac{1}{2b} \overset{*}{A}_{12} H^{122}_{133ijnmpq} \right.$$

$$\left. + \frac{1}{b} \overset{*}{A}_{66} (H^{212}_{133ijnmpq} \right]$$

$$- C_{nm} D_{pq} \left[\frac{b}{a^2} \overset{*}{A}_{11} H^{111}_{134ijnmpq} + \frac{1}{b} \overset{*}{A}_{12} H^{122}_{134ijnmpq} \right.$$

$$\left. + \frac{1}{b} \overset{*}{A}_{66} (H^{212}_{134ijnmpq} + H_{134ijnmpq}) \right]$$

$$- D_{nm} D_{pq} \left[\frac{b}{2a^2} \overset{*}{A}_{11} H^{111}_{144ijnmpq} + \frac{1}{2b} \overset{*}{A}_{12} H^{122}_{144ijnmpq} \right.$$

$$\left. + \frac{1}{b} \overset{*}{A}_{66} (H^{212}_{144ijnmpq} \right]$$

$$- w_o \left(\frac{b}{R} \overset{*}{A}_{12} G^{10}_{15ij} - w_o \left[\frac{b}{2a^2} \overset{*}{A}_{11} H^{111}_{155ij} + \frac{1}{2b} \overset{*}{A}_{12} H^{122}_{155ij} \right. \right.$$

$$\left. + \frac{1}{b} \overset{*}{A}_{66} H^{212}_{155ij} \right] \qquad \text{(A-9)}$$

$$\frac{\partial \Pi}{\partial B_{ij}} = 0$$

$$A_{nm} \left[\overset{*}{A}_{12} G^{12}_{12nmij} + \overset{*}{A}_{66} G^{21}_{12nmij} \right]$$

$$+ B_{nm} \left[\frac{a}{b} A^*_{22} G^{22}_{22ijnm} + \frac{b}{a} A^*_{66} G^{11}_{22ijnm} + \frac{I_s E_s}{a^3} K^{33}_{22ijnm} + \frac{A_f E_f}{b} K^{22}_{22ijnm} \right]$$

$$+ C_{nm} \left[\frac{a}{R} A^*_{22} G^{20}_{23ijnm} + \frac{w_o}{a} A^*_{12} H^{211}_{235ijnm} + \frac{a w_o}{b^2} A^*_{22} H^{222}_{235ijnm} \right.$$

$$\left. + \frac{w_o}{a} A^*_{66} \left(H^{121}_{235ijnm} + H^{112}_{235ijnm} \right) \right]$$

$$+ D_{nm} \left[\frac{a}{R} A^*_{22} G^{20}_{24ijnm} + \frac{w_o}{a} A^*_{12} H^{211}_{245ijnm} + \frac{a w_o}{b^2} A^*_{22} H^{222}_{245ijnm} \right.$$

$$\left. + \frac{w_o}{a} A^*_{66} \left(H^{112}_{245ijnm} + H^{121}_{245ijnm} \right) \right]$$

$$+ a_1 \left(a A^*_{12} F^{2}_{2ij} \right) + b_1 \left(b A^*_{66} F^{1}_{2ij} \right)$$

$$= P_{xy} J_{2ij} - C_{nm} C_{pq} \left[\frac{1}{2a} A^*_{12} H_{233ijnmpq} + \frac{a}{2b^2} A^*_{22} H_{233ijnmpq} \right.$$

$$\left. + \frac{1}{a} A^*_{66} H^{122}_{233ijnmpq} \right]$$

$$- C_{nm} D_{pq} \left[\frac{1}{a} A^*_{12} H^{211}_{234ijnmpq} + \frac{a}{b^2} A^*_{22} H^{222}_{234ijnmpq} \right.$$

$$\left. + \frac{1}{a} A^*_{66} \left(H^{112}_{234ijnmpq} + H^{121}_{234ijnmpq} \right) \right]$$

$$- D_{nm} D_{pq} \left[\frac{1}{2a} A^*_{12} H^{211}_{244ijnmpq} + \frac{a}{2b^2} A^*_{22} H^{222}_{244ijnmpq} \right.$$

$$\left. + \frac{1}{a} A^*_{66} H^{112}_{244ijnmpq} \right] - w_o \left(\frac{a}{R} A^*_{22} G^{20}_{25ij} \right)$$

$$- w_o \left[\frac{1}{2a} A^*_{12} H^{211}_{255ij} + \frac{a}{2b^2} A^*_{22} H^{222}_{255ij} + \frac{1}{a} A^*_{66} H^{112}_{255ij} \right] \qquad \text{(A-10)}$$

$$\frac{\partial \Pi}{\partial C_{ij}} = 0$$

$$A_{nm}\left[\frac{b}{R} A^*_{12}\, G^{10}_{13nmij} + \frac{bw_o}{a^2} A^*_{11}\, H^{111}_{135nmij} + \frac{w_o}{b} A^*_{12}\, H^{122}_{135nmij}\right.$$

$$\left. + \frac{w_o}{b} A^*_{66}\, (H^{221}_{135nmij} + H^{212}_{135nmij})\right]$$

$$+ B_{nm}\left[\frac{a}{R} A^*_{22}\, G^{20}_{23nmij} + \frac{aw_o}{b^2} A^*_{22}\, H^{222}_{235nmij} + \frac{w_o}{a} A^*_{12}\, H^{211}_{235nmij}\right.$$

$$\left. + \frac{w_o}{a} A^*_{66}\, (H^{121}_{235nmij} + H^{112}_{235nmij})\right]$$

$$+ C_{nm}\left[\frac{ab}{R^2} A^*_{22}\, G^{00}_{33ijnm} + \frac{b}{a3} D^*_{11}\, G^{33}_{33ijnm}\right.$$

$$+ \frac{D^*_{12}}{ab} (G^{34}_{33ijnm} + G^{34}_{33nmij}) + \frac{2}{a^2} D^*_{16}\, (G^{35}_{33ijnm} + G^{35}_{33nmij})$$

$$+ \frac{a}{b^3} D^*_{22}\, G^{44}_{33ijnm} + \frac{2}{b^2} D^*_{26}\, (G^{45}_{33ijnm} + G^{45}_{33nmij})$$

$$+ \frac{4D^*_{66}}{ab} G^{55}_{33ijnm} + \frac{bw_o}{aR} A^*_{12}\, (H^{011}_{335ijnm} + H^{011}_{335nmij} + H^{110}_{335ijnm})$$

$$+ \frac{aw_o}{bR} A^*_{22}\, (H^{022}_{335ijnm} + H^{220}_{335ijnm} + H^{022}_{335nmij})$$

$$+ \frac{3bw_o^2}{2a^3} A^*_{11}\, I^{1111}_{3355ijnm}$$

$$+ \frac{w_o^2}{2ab} (A^*_{12} + 2A^*_{66})\, (I^{1122}_{3355ijnm} + 2I^{1212}_{3355ijnm} + 2I^{1212}_{3355nmij}$$

$$\left. + I^{2211}_{3355ijnm}) + \frac{3aw_o^2}{2b^3} A^*_{22}\, I^{2222}_{3355ijnm}\right]$$

$$+ D_{nm} \left[\frac{ab}{R^2} A^*_{22} G^{00}_{34ijnm} + \frac{b}{a^3} D^*_{11} G^{33}_{34ijnm} \right.$$

$$+ \frac{D^*_{12}}{ab} (G^{34}_{34ijnm} + G^{43}_{34ijnm}) + \frac{2}{a^2} D^*_{16} (G^{35}_{34ijnm} + G^{53}_{34ijnm})$$

$$+ \frac{a}{b^3} D^*_{22} G^{44}_{34ijnm} + \frac{2}{b^2} D^*_{26} (G^{45}_{34ijnm} + G^{54}_{34ijnm})$$

$$+ \frac{4}{ab} D^*_{66} G^{55}_{34ijnm} + \frac{bw_o}{aR} A^*_{12} (H^{011}_{345ijnm} + H^{101}_{345ijnm} + H^{110}_{345ijnm})$$

$$+ \frac{aw_o}{bR} A^*_{22} (H^{022}_{345ijnm} + H^{202}_{345ijnm} + H^{220}_{345ijnm}) + \frac{3bw_o^2}{2a^3} A^*_{11} I^{1111}_{3455ijnm}$$

$$+ \frac{w_o^2}{2ab} (A^*_{12} + 2A^*_{66}) (I^{1122}_{3455ijnm} + 2I^{1212}_{3455ijnm} + 2I^{2112}_{3455ijnm} + I^{2211}_{3455ijnm})$$

$$\left. + \frac{3aw_o^2}{2b^3} A^*_{22} I^{2222}_{3455ijnm} \right]$$

$$+ a_1 \left[\frac{ab}{R} A^*_{12} F^{0}_{3ij} + \frac{bw_o}{a} A^*_{11} G^{11}_{35ij} + \frac{aw_o}{b} A^*_{12} G^{22}_{35ij} \right]$$

$$+ b_1 \left[w_o A^*_{66} (G^{21}_{35ij} + G^{12}_{35ij}) \right]$$

$$= - A_{nm} C_{pq} \left[\frac{b}{a^2} A^*_{11} H^{111}_{133nmijpq} + \frac{1}{b} A^*_{12} H^{122}_{133nmijpq} \right.$$

$$\left. + \frac{1}{b} A^*_{66} (H^{212}_{133nmijpq} + H^{212}_{133nmpqij}) \right]$$

$$- A_{nm} D_{pq} \left[\frac{b}{a^2} A^*_{11} H^{111}_{134nmijpq} + \frac{1}{b} A^*_{12} H^{122}_{134nmijpq} \right.$$

$$\left. + \frac{1}{b} A_{66} (H_{134nmijpq} + H_{134nmijpq}) \right]$$

$$- B_{nm} C_{pq} \Big[\frac{1}{a} A^*_{12} H^{211}_{233nmijpq} + \frac{a}{b^2} A^*_{22} H^{222}_{233nmijpq}$$

$$+ \frac{1}{a} A^*_{66} (H^{112}_{233nmijpq} + H^{112}_{233nmpqij}) \Big]$$

$$- B_{nm} D_{pq} \Big[\frac{1}{a} A^*_{12} H^{211}_{234nmijpq} + \frac{a}{b^2} A^*_{22} H^{222}_{234nmijpq}$$

$$+ \frac{1}{a} A^*_{66} (H^{112}_{234nmijpq} + H^{121}_{234nmijpq}) \Big]$$

$$- C_{nm} C_{pq} \Big[\frac{b}{2aR} A^*_{12} (H^{011}_{333ijnmpq} + 2H^{011}_{333nmijpq})$$

$$+ \frac{a}{2bR} A^*_{22} (H^{022}_{333ijnmpq} + 2H^{022}_{333nmijpq}) + \frac{3bw_o}{2a^3} A^*_{11} (I^{1111}_{3335ijnmpq})$$

$$+ \frac{w_o}{2ab} (A^*_{12} + 2A^*_{66}) (2I^{1122}_{3335ijnmpq} + I^{1122}_{3335nmpqij} + I^{1221}_{3335ijnmpq}$$

$$+ 2I^{1221}_{3335nmijpq}) + \frac{3aw_o}{2b^3} A^*_{22} I^{2222}_{3335ijnmpq} \Big]$$

$$- C_{nm} D_{pdq} \Big[\frac{b}{aR} A^*_{12} (H^{011}_{334ijnmpq} + H^{011}_{334nmijpq} + H^{110}_{334ijnmpq})$$

$$+ \frac{a}{bR} A^*_{22} (H^{022}_{334ijnmpq} + H^{220}_{334ijnmpq} + H^{022}_{334nmijpq})$$

$$+ \frac{3bw_o}{a^3} A^*_{11} I^{1111}_{3345ijnmpq} + \frac{w_o}{ab} (A^*_{12} + 2A^*_{66}) (I^{1122}_{3345ijnmpq} + I^{1212}_{3345ijnmpq}$$

$$+ I^{1212}_{3345nmijpq} + I^{1221}_{3345ijnmpq} + I^{1221}_{3345nmijpq} + I^{2211}_{3345ijnmpq})$$

$$+ \frac{3aw_o}{b^3} A^*_{22} I^{2222}_{3345ijnmpq} \Big]$$

$$- D_{nm} D_{pq} \left[\frac{b}{2aR} A_{12}^{*} (H_{344ijnmpq}^{011} + 2H_{344ijnmpq}^{110}) + \frac{a}{2bR} A_{22}^{*} H_{344ijnmpq}^{022} \right.$$

$$+ 2H_{344ijnmpq}^{202}) + \frac{3bw_o}{2a^3} A_{11}^{*} I_{3445ijnmpq}^{1111}$$

$$+ \frac{w_o}{2ab} (A_{12}^{*} + 2A_{66}^{*}) (2I_{3445ijnmpq}^{1122} + I_{3445ijnmpq}^{2112} + I_{3445ijnmpq}^{1221}$$

$$\left. + 2I_{3445ijnmpq}^{2211}) + \frac{3aw_o}{2b} A_{22}^{*} I_{3445ijnmpq}^{2222} \right]$$

$$- a_1 C_{nm} \left[\frac{b}{a} A_{11}^{*} G_{33ijnm}^{11} + \frac{a}{b} A_{12}^{*} G_{33ijnm}^{22} \right]$$

$$- a_1 D_{nm} \left[\frac{b}{a} A_{11}^{*} G_{34ijnm}^{11} + \frac{a}{b} A_{12}^{*} G_{34ijnm}^{22} \right]$$

$$- b_1 C_{nm} \left[A_{66}^{*} G_{33nmij}^{12} + G_{33ijnm}^{12}) \right]$$

$$- b_1 D_{nm} \left[A_{66}^{*} G_{34ijnm}^{12} + G_{34ijnm}^{21}) \right]$$

$$- C_{nm} C_{pq} C_{rs} \left[\frac{b}{2a^3} A_{11}^{*} I_{3333ijnmpqrs}^{1111} + \frac{1}{2ab} (A_{12}^{*} + 2A_{66}^{*})(I_{3333ijnmpqrs}^{1122} \right.$$

$$\left. + I_{3333nmpqijrs}^{1122}) + \frac{a}{2b^3} A_{22}^{*} I_{3333ijnmpqrs}^{2222} \right]$$

$$- C_{nm} C_{pq} D_{rs} \left[\frac{3b}{2a^3} A_{11}^{*} I_{3333ijnmpqrs}^{1111} + \frac{1}{2ab} (A_{12}^{*} + 2A_{66}^{*})(2I_{3334ijnmpqrs}^{1122} \right.$$

$$+ I_{3334nmpqijrs}^{1122} + I_{3334ijnmpqrs}^{1221} + 2I_{3334nmijpqrs}^{1221})$$

$$\left. + \frac{3a}{2b^3} A_{22}^{*} I_{3334ijnmpqrs}^{2222} \right]$$

$$- C_{nm} D_{pq} D_{rs} \Big[\frac{3b}{2a^3} A^*_{11} I^{1111}_{3344ijnmpqrs}$$

$$+ \frac{1}{2ab} (A^*_{12} + 2A^*_{66}) (I^{1122}_{3344ijnmpqrs} + 2I^{1212}_{3344ijnmpqrs} + 2I^{1212}_{3344nmijpqrs}$$

$$+ I^{2211}_{3344ijnmpqrs}) + \frac{3a}{2b^3} A^*_{22} I^{2222}_{3344ijnmpqrs} \Big]$$

$$- D_{nm} D_{pq} D_{rs} \Big[\frac{b}{2a^3} A^*_{11} I^{1111}_{3444ijnmpqrs}$$

$$+ \frac{1}{2ab} (A^*_{12} + 2A^*_{66}) (I^{1122}_{3444ijnmpqrs} + I^{2112}_{3444ijnmpqrs})$$

$$+ \frac{a}{2b^3} A^*_{22} I^{2222}_{3444ijnmpqrs} \Big]$$

$$- w_o \Big[\frac{ab}{R^2} A^*_{22} G^{00}_{35ij} + \frac{b}{a^3} D^*_{11} G^{33}_{35ij} + \frac{D^*_{12}}{ab} (G^{43}_{35ij} + G^{34}_{35ij})$$

$$+ \frac{2D^*_{16}}{a^2} (G^{35}_{35ij} + G^{53}_{35ij}) + \frac{a}{b^3} D^*_{22} G^{44}_{35ij} + \frac{2D^*_{26}}{b^2} (G^{45}_{35ij} + G^{54}_{35ij})$$

$$+ \frac{4D^*_{66}}{ab} G^{55}_{35ij} \Big]$$

$$- w_o^2 \Big[\frac{bA^*_{12}}{2aR} (H^{011}_{335ij} + 2H^{101}_{355ij}) + \frac{aA^*_{22}}{2bR} (H^{022}_{355ij} + 2H^{202}_{355ij}) \Big]$$

$$- w_o \Big[\frac{b}{2a^3} A^*_{11} I^{1111}_{3555ij} + \frac{1}{2ab} (A^*_{12} + 2A^*_{66})(I^{1122}_{3555ij} + I^{2112}_{3555ij})$$

$$+ \frac{a}{2b^3} A^*_{22} I^{2222}_{3555ij} \Big]$$

$$\frac{\partial \Pi}{\partial D_{ij}} = 0$$

$$A_{nm} \left[\frac{b}{R} A^{*}_{12}\, G^{10}_{14nmij} + \frac{bw_o}{a^2} A^{*}_{11} + H^{111}_{145nmij} \right.$$

$$\left. + \frac{w_o}{b} A^{*}_{12}\, H^{122}_{145nmij} + \frac{w_o}{b} A^{*}_{66} \left(H^{212}_{145nmij} + H^{221}_{145nmij}\right) \right]$$

$$+ B_{nm} \left[\frac{a}{R} A^{*}_{22}\, G^{20}_{24nmij} + \frac{w_o}{a} A^{*}_{12}\, H^{211}_{245nmij} \; \frac{aw_o}{b^2} A^{*}_{22}\, H^{222}_{245nmij} \right.$$

$$\left. + \frac{w_o}{a} A^{*}_{66} \left(H^{112}_{245nmij} + (H^{121}_{245nmij}\right) \right]$$

$$+ C_{nm} \left[\frac{ab}{R^2} A^{*}_{22}\, G^{00}_{34nmij} + \frac{b}{a^3} D^{*}_{11}\, G^{33}_{34nmij} + \frac{D^{*}_{12}}{ab} \left(G^{34}_{34nmij} + G^{43}_{34nmij}\right) \right.$$

$$+ \frac{2D^{*}_{16}}{a^2} \left(G^{35}_{34nmij} + G^{53}_{34nmij}\right) + \frac{a}{b^3} D^{*}_{22}\, G^{44}_{34nmij}$$

$$+ \frac{2D^{*}_{26}}{b^2} \left(G^{45}_{34nmij} + G^{54}_{34nmij}\right) + \frac{4D^{*}_{66}}{ab}\, G^{55}_{34nmij}$$

$$+ \frac{bw_o}{aR} A^{*}_{12} \left(H^{011}_{345nmij} + H^{101}_{345nmij} + H^{110}_{345nmij}\right)$$

$$+ \frac{aw_o}{bR} A^{*}_{22} \left(H^{022}_{345nmij} + H^{202}_{345nmij} + H^{220}_{345nmij}\right)$$

$$+ \frac{3bw_o^2}{2a^3} A^{*}_{11}\, I^{1111}_{3455nmij} + \frac{w_o^2}{2ab} \left(A^{*}_{12} + 2A^{*}_{66}\right) \left(I^{1122}_{3455nmij}\right.$$

$$\left. \left. + 2I^{1212}_{3455nmij} + 2I^{2112}_{3455nmij} + I^{22113}_{3455nmij}\right) + \frac{3aw_o^2}{2b^3} A_{22}\, I^{2222}_{3455nmij} \right]$$

$$+ D_{nm} \left[\frac{ab}{R^2} A^{*}_{22}\, G^{00}_{44ijnm} + \frac{b}{a^3} D^{*}_{11}\, G^{33}_{44ijnm} + \frac{D^{*}_{12}}{ab} \left(G^{34}_{44ijnm} + G^{34}_{44nmij}\right) \right.$$

$$+ \frac{2D^*_{16}}{a^2} \left(G^{35}_{44ijnm} + G^{35}_{44nmij}\right) + \frac{aD^*_{22}}{b^3} G^{44}_{44ijnm} + \frac{2D^*_{26}}{b^2} \Big(G^{44}_{44ijnm}$$

$$+ G^{45}_{44nmij}\Big) + \frac{4D^*_{66}}{ab} G^{55}_{44ijnm} + \frac{bw_o}{aR} A^*_{12} \Big(H^{011}_{445ijnm} + H^{011}_{445nmij}$$

$$+ H^{110}_{445ijnm}\Big) + \frac{aw_o}{bR} A^*_{22} \left(H^{022}_{445ijnm} + H^{022}_{445nmij} + H^{220}_{445ijnm}\right)$$

$$+ \frac{3bw_o^2}{2a^3} A^*_{11} I^{1111}_{4455ijnm} + \frac{w_o^2}{2ab} \left(A^*_{12} + 2A^*_{66}\right)\Big(I^{1122}_{4455ijnm}$$

$$+ 2I^{1212}_{4455ijnm} + 2I^{1212}_{4455nmij} + I^{2211}_{4455ijnm}\Big) + \frac{3aw_o^2}{2b^3} A^*_{22} I^{2222}_{4455ijnm} \Bigg]$$

$$+ a_1 \left[\frac{ab}{R} A^*_{12} F^{0}_{4ij} + \frac{bw_o}{a} A^*_{11} G^{11}_{45ij} + \frac{aw_o}{b} A^*_{12} G^{22}_{45ij} \right]$$

$$+ b_1 \left[w_o A^*_{66} \left(G^{21}_{45ij} + G^{12}_{45ij}\right) \right]$$

$$= - A_{nm} C_{pq} \Bigg[\frac{b}{a^2} A^*_{11} H^{111}_{134nmpqij} + \frac{1}{b} A^*_{12} H^{122}_{134nmpqij}$$

$$+ \frac{1}{b} A^*_{66} \left(H^{212}_{134nmpqij} + H^{221}_{134nmpqij}\right) \Bigg]$$

$$- A_{nm} D_{pq} \Bigg[\frac{b}{a^2} A^*_{11} H^{111}_{144nmijpq} + \frac{1}{b} A^*_{12} H^{122}_{144nmijpq}$$

$$+ \frac{1}{b} A^*_{66} \left(H^{212}_{144nmijpq} + H^{212}_{144nmpqij}\right) \Bigg]$$

$$- B_{nm} C_{pq} \Bigg[\frac{1}{a} A^*_{12} H^{211}_{234nmpqij} + \frac{a}{b^2} A^*_{22} H^{222}_{234nmpqij}$$

$$+ \frac{1}{a} A^*_{66} \left(H^{112}_{234nmpqij} + H^{121}_{234nmpqij}\right) \Bigg]$$

$$- B_{nm} D_{pq} \left[\frac{1}{a} A^*_{12} H^{211}_{244nmijpq} + \frac{a}{b^2} A^*_{22} H^{222}_{244nmijpq} \right.$$

$$\left. + \frac{1}{a} A^*_{66} \left(H^{112}_{244nmijpq} + H^{112}_{244nmpqij} \right) \right]$$

$$- C_{nm} C_{pq} \left[\frac{b}{2aR} A^*_{12} \left(2H^{011}_{334nmpqij} + H^{110}_{334nmpqij} \right) \right.$$

$$+ \frac{a}{2bR} A^*_{22} \left(2H^{022}_{334nmpqij} + H^{220}_{334nmpqij} \right) + \frac{3bw_o}{2a^3} A^*_{11} I^{1111}_{3345nmpqij}$$

$$+ \frac{w_o}{2ab} \left(A^*_{12} + 2A^*_{66} \right) \left(I^{1122}_{3345nmpqij} + 2I^{1212}_{3345nmpqij} \right.$$

$$\left. \left. + 2I^{1221}_{3345nmpqij} + I^{2211}_{3345nmpqij} \right) + \frac{3aw_o}{2b^3} A^*_{22} I^{2222}_{3345nmpqij} \right]$$

$$- C_{nm} D_{pq} \left[\frac{b}{aR} A^*_{12} \left(H^{011}_{344nmijpq} + H^{110}_{344nmijpq} + H^{110}_{344nmpqij} \right) \right.$$

$$+ \frac{a}{bR} A^*_{22} \left(H^{022}_{344nmijpq} + H^{202}_{344nmijpq} + H^{202}_{344nmpqij} \right)$$

$$+ \frac{3bw_o}{a^3} A^*_{11} I^{1111}_{3445nmijpq} + \frac{w_o}{ab} \left(A^*_{12} + 2A^*_{66} \right) \left(I^{1122}_{3445nmijpq} \right.$$

$$+ I^{1122}_{3445nmpqij} + I^{2112}_{3445nmijpq} + I^{1221}_{3445nmijpq} + I^{2211}_{3445nmijpq}$$

$$\left. \left. + I^{2211}_{3445nmpqij} \right) + \frac{3aw_o}{b^3} A^*_{22} I^{2222}_{3445nmijpq} \right]$$

$$- D_{nm} D_{pq} \left[\frac{b}{2aR} A^*_{12} \left(H^{011}_{444ijnmpq} + 2H^{011}_{444nmijpq} \right) + \frac{a}{2bR} A^*_{22} \left(H^{022}_{444ijnmpq} \right. \right.$$

$$\left. + 2H^{022}_{444nmijpq} \right) + \frac{3bw_o}{2a^3} A^*_{11} I^{1111}_{4445ijnmpq}$$

$$+ \frac{w_o}{2ab} (A^*_{12} + 2A^*_{66}) (2I^{1122}_{4445ijnmpq} + I^{1122}_{4445nmpqij} + I^{1221}_{4445ijnmpq}$$

$$+ 2I^{1221}_{4445nmijpq}) + \frac{3aw_o}{2b^3} A^*_{22} I^{2222}_{4445ijnmpq} \Big]$$

$$- a_1 C_{nm} \Big[\frac{b}{a} A^*_{11} G^{11}_{34nmij} + \frac{a}{b} A^*_{12} G^{22}_{34nmij} \Big]$$

$$- a_1 D_{nm} \Big[\frac{b}{a} A^*_{11} G^{11}_{44ijnm} + \frac{a}{b} A^*_{12} G^{22}_{44ijnm} \Big]$$

$$- b_1 C_{nm} \Big[A^*_{66} (G^{21}_{34ijnm} + G^{21}_{34nmij}) \Big]$$

$$- b_1 D_{nm} \Big[A^*_{66} (G^{12}_{44ijnm} + G^{12}_{44nmij}) \Big]$$

$$- C_{nm} C_{pq} C_{rs} \Big[\frac{b}{2a^3} A^*_{11} I^{1111}_{3334nmpqrsij} + \frac{1}{2ab} (A^*_{12} + 2A_{66})(I^{1122}_{3334nmpqrsij}$$

$$+ I^{1221}_{3334nmpqrsij}) + \frac{a}{ab^3} A^*_{22} I^{2222}_{3334nmpqrsij} \Big]$$

$$- C_{nm} C_{pq} D_{rs} \Big[\frac{3b}{2a^3} A^*_{11} I^{1111}_{3344nmpqijrs} + \frac{1}{2ab} (A^*_{12} + 2A^*_{66})(I^{1122}_{3344nmpqijrs}$$

$$+ 2I^{1212}_{3344nmpqijrs} + 2I^{1212}_{3344nmpqrsij} + I^{2211}_{3344nmpqijrs})$$

$$+ \frac{3a}{2a^3} A^*_{22} I^{2222}_{3344nmpqijrs} \Big]$$

$$- C_{nm} D_{pq} D_{rs} \Big[\frac{3b}{2a^3} A^*_{11} I^{1111}_{3444nmijpqrs}$$

$$+ \frac{1}{2ab} (A^*_{12} + 2A^*_{66})(I^{1122}_{3444nmijpqrs} + 2I^{1122}_{3444nmpqijrs} + 2I^{2112}_{3444nmijpqrs}$$

$$+ I^{2112}_{3444nmpqrsij}) + \frac{3a}{2b^3} A^*_{22} I^{2222}_{3444nmijpqrs} \Big]$$

$$- D_{nm} D_{pq} D_{rs} \Big[\frac{b}{2a^3} A^*_{11} I^{1111}_{4444ijnmpqrs}$$

$$+ \frac{1}{2ab} (A^*_{12} + 2A^*_{66})(I^{1122}_{4444ijnmpqrs} + I^{1122}_{4444nmpqijrs})$$

$$+ \frac{a}{2b^3} A^*_{22} I^{2222}_{4444ijnmpqrs} \Big]$$

$$- w_o \Big[\frac{ab}{R^2} A^*_{22} G^{00}_{45ij} + \frac{b}{a^3} D^*_{11} G^{33}_{45ij} + \frac{D^*_{12}}{ab} (G^{34}_{45ij} + G^{43}_{45ij})$$

$$+ \frac{2D^*_{16}}{a^2} (G^{35}_{45ij} + G^{53}_{45ij}) + \frac{a}{b^3} D^*_{22} G^{44}_{45ij} + \frac{2D^*_{26}}{b^2} (G^{45}_{45ij} + G^{54}_{45ij})$$

$$+ \frac{4}{ab} D^*_{66} G^{55}_{45ij} \Big]$$

$$- w_o^2 \Big[\frac{b}{2aR} A^*_{12} (H^{011}_{455ij} + 2H^{110}_{455ij}) + \frac{a}{2bR} A^*_{22} (H^{022}_{455ij} + 2H^{202}_{455ij}) \Big]$$

$$- w_o^3 \Big[\frac{b}{2a^3} A^*_{11} I^{1111}_{4555ij} + \frac{1}{2ab} (A^*_{12} + 2A^*_{66}) (I^{1122}_{4555ij} + I^{2112}_{4555ij})$$

$$+ \frac{a}{ab^3} A^*_{22} I^{2222}_{4555ij} \Big]$$

$$\frac{\partial \Pi}{\partial a_1} = 0$$

$$A_{nm} \Big[b A^*_{11} F^1_{1nm} + A_s E_s J^1_{1nm} \Big]$$

$$+ B_{nm} \Big[a A^*_{12} F^2_{2nm} \Big]$$

$$+ C_{nm} \left[\frac{ab}{R} \overset{*}{A}_{12} F^{0}_{3nm} + \frac{bw_o}{a} \overset{*}{A}_{11} G^{11}_{35nm} + \frac{aw_o}{b} \overset{*}{A}_{12} G^{22}_{35nm} \right]$$

$$+ D_{nm} \left[\frac{ab}{R} \overset{*}{A}_{12} F^{0}_{4nm} + \frac{bw_o}{a} \overset{*}{A}_{11} G^{11\,*}_{45nm} + \frac{aw_o}{b} \overset{*}{A}_{12} G^{22}_{45nm} \right]$$

$$+ a_1 (ab \overset{*}{A}_{11} + a A_s E_s)$$

$$= P_{xx}\, a - C_{nm} C_{pq} \left[\frac{b}{2a} \overset{*}{A}_{11} G^{11}_{33nmpq} + \frac{a}{2b} \overset{*}{A}_{12} G^{22}_{33nmpq} \right]$$

$$- C_{nm} D_{pq} \left[\frac{b}{a} \overset{*}{A}_{11} G^{11}_{34nmpq} + \frac{a}{b} \overset{*}{A}_{12} G^{22}_{34nmpq} \right]$$

$$- D_{nm} D_{pq} \left[\frac{b}{2a} \overset{*}{A}_{11} G^{11}_{44nmpq} + \frac{a}{ab} \overset{*}{A}_{12} G^{22}_{44nmpq} \right]$$

$$- \frac{abw_o}{R} \overset{*}{A}_{12} F^{0}_{5} - \frac{bw^2_o}{2a} \overset{*}{A}_{11} G^{11}_{55} - \frac{aw^2_o}{2b} \overset{*}{A}_{12} G^{22}_{55}$$

$$\frac{\partial \Pi}{\partial b_1} = 0$$

$$A_{nm} (a \overset{*}{A}_{66} F^{2}_{1nm})$$

$$+ B_{nm} (b \overset{*}{A}_{66} F^{1}_{2nm})$$

$$+ C_{nm} \left[w_o \overset{*}{A}_{66} (G^{21}_{35nm} + G^{12}_{35nm}) \right]$$

$$+ D_{nm} \left[w_o \overset{*}{A}_{66} (G^{12}_{45nm} + G^{21}_{45nm}) \right]$$

$$+ b_1 \left[ab \, A_{66}^{*} \right]$$

$$= a \, P_{xy} - A_{66}^{*} \Big[C_{nm} \, C_{pq} \, G^{12}_{33nmpq} + C_{nm} \, D_{pq} \, (G^{12}_{34nmpq} + G^{21}_{34nmpq})$$

$$+ D_{nm} \, D_{pq} \, G^{12}_{44nmpq} + w_o \, G^{12}_{55} \Big]$$

The individual energy integrals defined in Equations (A-2) through (A-8) are evaluated in closed form for the particular displacement functions given by Equations (59). These integrals are expressed in terms of simple integrals of the sine and cosine functions and the combinations of those functions. The assumed displacement functions are given in equations (59) in Section 3 and they are rewritten below.

$$f_1(n,m;x,y) = X_2\ (1,n)\ \cos m\pi y + X_1\ (1,n)\ \sin m\pi y$$

$$f_2(n,m;x,y) = [\cos n\pi x - X_3(1,n)]\ \cos m\pi y + [\sin n\pi x - X_2(n,1)]\ \sin m\pi y$$

$$f_3(n,m;x,y) = \sin n\pi x\ \sin m\pi y$$

$$f_4(n,m,x,y) = X_1(1,n)\ Y_2(1,m) - X_2(1,n)\ Y_1(1,m)$$

$$f_5(x,y) = \sin \pi x\ \sin \pi y$$

where

$$X_1(n,m) = \sin n\pi x\ \sin m\pi x, \quad Y_1(n,m) = \sin n\pi y\ \sin m\pi y$$

$$X_2(n,m) = \sin n\pi x\ \cos m\pi x, \quad Y_2(n,m) = \sin n\pi y\ \cos m\pi y$$

$$X_3(n,m) = \cos n\pi x\ \cos m\pi x, \quad Y_3(n,m) = \cos n\pi y\ \cos m\pi y$$

The derivatives of the displacement functions are given by

$$f_{1,1}\ (n,m;x,y) = \pi[Z_1(n)\cos m\pi y + Z_2(n)\ \sin m\pi y]$$

$$f_{1,2}\ (n,m;x,y) = m\pi[X_1(1,n)\cos m\pi y - X_2(1,n)\ \sin m\pi y]$$

$$f_{2,1}\ (n,m;x,y) = \pi\{[n\ X_2(n,1) + X_2(1,n) - n\ \sin n\pi x]\ \cos m\pi y$$
$$+ [n\ \cos n\pi x - n\ X_3(1,n) + X_1(1,n)]\ \sin m\pi y\}$$

$$f_{2,2}\ (n,m;x,y) = m\pi\{[\sin n\pi x - X_2(n,1)]\ \cos m\pi y$$
$$- [\cos n\pi x - X_3(1,n)]\ \sin m\pi y\}$$

$$f_{3,1}(n,m;x,y) = n\pi \cos n\pi x \sin m\pi y$$

$$f_{3,2}(n,m;x,y) = m\pi \sin n\pi x \cos m\pi y$$

$$f_{3,3}(n,m;x,y) = -n^2\pi^2 \sin n\pi x \sin m\pi y$$

$$f_{3,4}(n,m;x,y) = -m^2\pi^2 \sin n\pi x \sin m\pi y$$

$$f_{3,5}(n,m;x,y) = nm\pi^2 \cos n\pi x \cos m\pi y$$

$$f_{4,1}(n,m;x,y) = \pi[Z_2(n)Y_2(1,m) - Z_1(n)\, Y_1(1,m)]$$

$$f_{4,2}(n,m;x,y) = \pi[X_1(1,n)\bar{Z}_1(m) - X_2(1,n)\, \bar{Z}_2(m)]$$

$$f_{4,3}(n,m;x,y) = \pi^2\{[2n\, X_3(1,n) - (1+n^2)\, X_1(1,n)]\, Y_2(1,m) + [2n\, X_2(n,1) + (1+n^2)\, X_2(1,n)]\, Y_1(1,m)\}$$

$$f_{4,4}(n,m;x,y) = -\pi^2\{X_1(1,n)[(1+m^2)Y_2(1,m) + 2m\, Y_2(m,1)] + X_2(1,n)[2m\, Y_3(1,m) - (1+m^2)\, Y_1(1,m)]\}$$

$$f_{4,5}(n,m;x,y) = \pi^2[Z_2(n)\bar{Z}_1(m) - Z_1(n)\, \bar{Z}_2(m)]$$

$$f_{5,1}(x,y) = \pi \cos \pi x \sin \pi y$$

$$f_{5,2}(x,y) = \pi \sin \pi x \cos \pi y$$

$$f_{5,3}(x,y) = -\pi^2 \sin \pi x \sin \pi y$$

$$f_{5,4}(x,y) = -\pi^2 \sin \pi x \sin \pi y$$

$$f_{5,5}(x,y) = \pi^2 \cos \pi x \cos \pi y$$

where,

$$Z_1(n) = X_3(1,n) - n\, X_1(1,n) \quad , \quad \bar{Z}_1(n) = Y_3(1,n) - n\, Y_1(1,n)$$

$$Z_2(n) = X_2(n,1) + n\, X_2(1,n) \quad , \quad \bar{Z}_2(n) = Y_2(n,1) + n\, Y_2(1,n)$$

The integrals are expressed in terms of a series of elementary integrals and combination of these elementary integrals defined below.

$$I(n) = \int_0^1 \cos n\pi x\, dx = \begin{cases} 0 & \text{for} \quad n \neq 0 \\ 1 & \text{for} \quad n = 0 \end{cases}$$

$$J(n) = \int_0^1 \sin n\pi x dx = \begin{cases} 0 & \text{for even } n \\ \frac{2}{n\pi} & \text{for odd } n \end{cases}$$

$$I_0(n,m) = \int_0^1 \sin n\pi x \sin m\pi x dx = \frac{1}{2} [I(n-m) - I(n+m)]$$

$$J_1(n,m) = \int_0^1 \sin n\pi x \cos m\pi x dx = \frac{1}{2} [J(n-m) - J(n+m)]$$

$$I_2(n,m) = \int_0^1 \cos n\pi x \cos m\pi x dx = \frac{1}{2} [I(n-m) + I(n+m)]$$

$$J_0(n,m,p) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x dx$$

$$= \frac{1}{4} [J(n-m+p) + J(n+m-p) - J(n-m-p) - J(n+m+p)]$$

$$I_1(n,m,p) = \int_0^1 \sin n\pi x \sin m\pi x \cos p\pi x dx$$

$$= \frac{1}{4} [I(n-m+p) - I(n+m-p) + I(n-m-p) - I(n+m+p)]$$

$$J_2(n,m,p) = \int_0^1 \sin n\pi x \cos m\pi x \cos p\pi x dx$$

$$= \frac{1}{4} [J(n-m+p) + J(n+m-p) + J(n-m-p) + J(n+m+p)]$$

$$I_3(n,m,p) = \int_0^1 \cos n\pi x \cos m\pi x \cos p\pi x dx$$

$$= \frac{1}{4} [I(n-m+p) + I(n+m-p) + I(n-m-p) + I(n+m+p)]$$

$$I_0(n,m,p,q) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x dx$$

$$= \frac{1}{2} [I_1(n,m,p-q) - I_1(n,m,p+q)]$$

$$J_1(n,m,p,q) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \cos q\pi x dx$$

$$= \frac{1}{2} [J_0(n,m,p-q) + J_0(n,m,p+q)]$$

$$I_2(n,m,p,q) = \int_0^1 \sin n\pi x \sin m\pi x \cos p\pi x \cos q\pi x dx$$

$$= \frac{1}{2} [I_1(n,m,p-q) + I_1(n,m,p+q)]$$

$$J_3(n,m,p,q) = \int_0^1 \sin n\pi x \cos m\pi x \cos p\pi x \cos q\pi x dx$$

$$= \frac{1}{2} [J_2(n,m,p-q) + J_2(n,m,p+q)]$$

$$I_4(n,m,p,q) = \int_0^1 \cos n\pi x \cos m\pi x \cos p\pi x \cos q\pi x dx$$

$$= \frac{1}{2} [I_3(n,m,p-q) + I_3(n,m,p+q)]$$

$$J_0(n,m,p,q,r) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x \sin r\pi x dx$$

$$= \frac{1}{2} [J_1(n,m,p,q-r) - J_1(n,m,p,q+r)]$$

$$I_1(n,m,p,q,r) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x \cos r\pi x dx$$

$$= \frac{1}{2} [I_0(n,m,p,q-r) + I_0(n,m,p,q+r)]$$

$$J_2(n,m,p,q,r) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \cos q\pi x \cos r\pi x dx$$

$$= \frac{1}{2} [J_1(n,m,p,q-r) + J_1(n,m,p,q+r)]$$

$$I_3(n,m,p,q,r) = \int_0^1 \sin n\pi x \sin m\pi x \cos p\pi x \cos q\pi x \cos r\pi x dx$$

$$= \frac{1}{2} [I_2(n,m,p,q-r) + I_2(n,m,p,q+r)]$$

$$J_4(n,m,p,q,r) = \int_0^1 \sin n\pi x \cos m\pi x \cos p\pi x \cos q\pi x \cos r\pi x dx$$

$$= \frac{1}{2} [J_3(n,m,p,q-r) + J_3(n,m,p,q+r)]$$

$$I_5(n,m,p,q,r) = \int_0^1 \cos n\pi x \cos m\pi x \cos p\pi x \cos q\pi x \cos r\pi x dx$$

$$= \frac{1}{2} [I_4(n,m,p,q-r) + I_4(n,m,p,q+r)]$$

$$I_0(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x \sin r\pi x \sin s\pi x dx$$

$$= \frac{1}{2} [I_1(n,m,p,q,r-s) - I_1(n,m,p,q,r+s)]$$

$$J_1(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x \sin r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [J_0(n,m,p,q,r-s) + J_0(n,m,p,q,r+s)]$$

$$I_2(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \sin q\pi x \cos r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [I_1(n,m,p,q,r-s) + I_1(n,m,p,q,r+s)]$$

$$J_3(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \sin m\pi x \sin p\pi x \cos q\pi x \cos r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [J_2(n,m,p,q,r-s) + J_2(n,m,p,q,r+s)]$$

$$I_4(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \sin m\pi x \cos p\pi x \cos q\pi x \cos r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [I_3(n,m,p,q,r-s) + I_3(n,m,p,q,r+s)]$$

$$J_5(n,m,p,q,r,s) = \int_0^1 \sin n\pi x \cos m\pi x \cos p\pi x \cos q\pi x \cos r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [J_4(n,m,p,q,r-s) + J_4(n,m,p,q,r+s)]$$

$$I_6(n,m,p,q,r,s) = \int_0^1 \cos n\pi x \cos m\pi x \cos p\pi x \cos q\pi x \cos r\pi x \cos s\pi x dx$$

$$= \frac{1}{2} [I_5(n,m,p,q,r-s) + I_5(n,m,p,q,r+s)]$$

In addition to the elementary integrals defined above, the following combinations of elementary integrals are also defined.

$$B_1(i,j,k,\iota) = \int_0^1 \cos i\pi x \cos j\pi x \; Z_2(k) \; Z_2(\iota)dx$$

$$= I_4(k,\iota,i,j,1,1) + kI_4(1,\iota,i,j,k,1)$$

$$+\iota I_4(1,k,i,j,\iota,1) + k\iota I_4(1,1,i,j,k,\iota)$$

$$B_2(i,j,k,\iota) = \int_0^1 \cos i\pi x \cos j\pi x \; Z_1(k) \; Z_2(\iota)dx$$

$$= J_5(\iota,i,j,1,1) - kJ_3(1,k,\iota,i,j,1)$$

$$+\iota J_5(1,i,j,k,\iota,1) - k\iota J_3(1,1,k,i,j,\iota)$$

$$B_3(i,j,k,\iota) = \int_0^1 \cos i\pi x \cos j\pi x \; Z_1(k) \; Z_1(\iota)dx$$

$$= I_6(i,j,k,\iota,1,1) - kI_4(1,k,i,j,\iota,1)$$

$$-\iota I_4(1,\iota,i,j,k,1) + k\iota I_2(1,1,k,\iota,i,j)$$

$$B_4(i,j,k,\iota) = \int_0^1 \sin i\pi x \sin j\pi x \; Z_1(k) \; Z_1(\iota)dx$$

$$= I_4(i,j,k,\iota,1,1) - kI_2(1,i,j,k,\iota,1)$$

$$-\iota I_2(1,i,j,\iota,k,1) + k\iota I_0(1,1,i,j,k,\iota)$$

$$B_5(i,j,k,\iota) = \int_0^1 \sin i\pi x \sin j\pi x \; Z_1(k) \; Z_2(\iota)dx$$

$$= J_3(i,j,\iota,k,1,1) - kJ_1(1,i,j,k,\iota,1)$$

$$+\iota J_3(1,i,j,k,\iota,1) - k\iota J_1(1,1,i,j,k,\iota)$$

$$B_6(i,j,k,\iota) = \int_0^1 \sin i\pi x \sin j\pi x\, Z_2(k)\, Z_2(\iota)dx$$

$$= I_2(i,j,k,\iota,1,1) + kI_2(1,i,j,\iota,k,1)$$

$$+\iota I_2(1,i,j,k,\iota,1) + k\iota I_2(1,1,i,j,k,\iota)$$

$$C_1(i,j,k,\iota) = \int_0^1 \sin i\pi x \cos j\pi x\, X_1(1,k)\, Z_2(\iota)dx$$

$$= I_2(1,i,k,\iota,j,1) + \iota I_2(1,1,i,k,j,\iota)$$

$$C_2(i,j,k,\iota) = \int_0^1 \sin i\pi x \cos j\pi x\, X_1(1,k)\, Z_1(\iota)dx$$

$$= J_3(1,i,k,j,\iota,,1) - \iota J_1(1,1,i,k,\iota,j)$$

$$C_3(i,j,k,\iota) = \int_0^1 \sin i\pi x \cos j\pi x\, X_2(1,k)\, Z_2(\iota)dx$$

$$= J_3(1,i,\iota,j,k,1) + \iota J_3(1,i,k,j,\iota,1)$$

$$C_4(i,j,k,\iota) = \int_0^1 \sin i\pi x \cos j\pi x\, X_2(1,k)\, Z_1(\iota)dx$$

$$= I_4(1,i,j,k,\iota,1) - \iota I_2(1,1,i,\iota,j,k)$$

$$D_1(i,j,k,\iota) = \int_0^1 \sin i\pi x\, X_2(1,j)\, X_2(1,k)\, X_2(1,\iota)dx$$

$$= \frac{1}{2}\,[I_3(1,i,j,k,\iota) - I_4(1,i,j,k,\iota,2)]$$

$$D_2(i,j,k,\iota) = \int_0^1 \sin i\pi x\, X_1(1,j)\, X_2(1,k)\, X_2(1,\iota)dx$$

$$= \frac{1}{2}\,[J_2(1,i,j,k,\iota) - J_3(1,i,j,k,\iota,2)]$$

$$D_3(i,j,k,\ell) = \int_0^1 \sin i\pi x \; X_1(1,j) \; X_1(1,k) \; X_2(1,\ell)dx$$

$$= \frac{1}{2} [I_1(1,i,j,k,\ell) - I_2(1,i,j,k,\ell,2)]$$

$$D_4(i,j,k,\ell) = \int_0^1 \sin i\pi x \; X_1(1,j) \; X_1(1,k) \; X_1(1,\ell)dx$$

$$= \frac{1}{2} [J_0(1,i,j,k,\ell) - J_1(1,i,j,k,\ell,2)]$$

$$E_1(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_2(j) \; Z_2(k) \; Z_2(\ell)dx$$

$$= \frac{1}{2} \{J_2(j,k,\ell,i,1) + J_3(j,k,\ell,i,1,2) + j \; J_3(2,k,\ell,i,j,1)$$

$$+ k \; J_3(2,j,\ell,i,k,1) + \ell \; J_3(2,j,k,i,\ell,1) + jk \; J_3(1,2,\ell,i,j,k)$$

$$+ j\ell \; J_3(1,2,k,i,j,\ell) + k\ell \; J_3(1,2,j,i,k,\ell)$$

$$+ jk\ell \; [J_4(1,i,j,k,\ell) - J_5(1,i,j,k,\ell,2)]\}$$

$$E_2(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_1(j) \; Z_2(k) \; Z_2(\ell)dx$$

$$= \frac{1}{2} \{I_3(k,\ell,i,j,1) + I_4(k,\ell,i,j,1,2) - j \; I_2(2,j,k,\ell,i,1)$$

$$+ k \; I_4(2,\ell,i,j,k,1) + \ell \; I_4(2,k,i,j,\ell,1) - jk \; I_2(1,2,j,\ell,i,k)$$

$$- j\ell \; I_2(1,2,j,k,i,\ell) + k\ell \; I_4(1,2,i,j,k,\ell)$$

$$- jk\ell \; [I_3(1,j,i,k,\ell) - I_4(1,j,i,k,\ell,2)]\}$$

$$E_3(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_1(j) \; Z_1(k) \; Z_2(\ell)dx$$

$$= \frac{1}{2} \{J_4(\ell,i,j,k,1) + J_5(\ell,i,j,k,1,2) - j \; J_3(2,j,\ell,i,k,1)$$

$$- k \; J_3(2,k,\ell,i,j,1) + \ell \; J_5(2,i,j,k,\ell,1) + jk \; J_1(1,2,j,k,\ell,i,)$$

$$- j\ell \; J_3(1,2,j,i,k,\ell) - k\ell \; J_3(1,2,k,i,j,\ell)$$

$$+ jk\ell \; [J_2(1,j,k,i,\ell) - J_3(1,j,k,i,\ell,2)]\}$$

$$E_4(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_1(j)\; Z_1(k)\; Z_1(\ell)dx$$

$$= \frac{1}{2} \{I_5(i,j,k,\ell,1) + I_6(i,j,k,\ell,1,2) - j\; I_4(2,j,i,k,\ell,1)$$

$$- k\; I_4(2,k,i,j,\ell,1) - \ell\; I_4(2,\ell,i,j,k,1) + jk\; I_2(1,2,j,k,i,\ell)$$

$$+ j\ell\; I_2(1,2,j,\ell,i,k) + k\ell\; I_2(1,2,k,\ell,i,j)$$

$$- jk\ell\; [I_1(1,j,k,\ell,i) - I_2(1,j,k,\ell,i,2)]\}$$

$$F_1(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_2(j)\; X_1(1,k)\; X_1(1,\ell)dx$$

$$= \frac{1}{2} \{J_1(1,2,j,k,\ell,i) + j[J_2(1,k,\ell,i,j) - J_3(1,k,\ell,i,j,2)]\}$$

$$F_2(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_2(j)\; X_1(1,k)\; X_2(1,\ell)dx$$

$$= \frac{1}{2} \{I_2(1,2,j,k,i,\ell) + j[I_3(1,k,i,j,\ell) - I_4(1,k,i,j,\ell,2)]\}$$

$$F_3(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_2(j)\; X_2(1,k)\; X_2(1,\ell)dx$$

$$= \frac{1}{2} \{J_3(1,2,j,i,k,\ell) + j[J_4(1,i,j,k,\ell) - J_5(1,i,j,k,\ell,2)]\}$$

$$F_4(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_1(j)\; X_1(1,k)\; X_1(1,\ell)dx$$

$$= \frac{1}{2} \{I_2(1,2,k,\ell,i,j) - j[I_1(1,j,k,\ell,i) - I_2(1,j,k,\ell,i,2)]\}$$

$$F_5(i,j,k,\ell) = \int_0^1 \cos i\pi x \; Z_1(j)\; X_1(1,k)\; X_2(1,\ell)dx$$

$$= \frac{1}{2} \{J_3(1,2,k,i,j,\ell) - j[J_2(1,j,k,i,\ell) - J_3(1,j,k,i,\ell,2)]\}$$

$$F_6(i,j,k,\iota) = \int_0^1 \cos i\pi x \; Z_1(j) \; X_2(1,k) \; X_2(1,\iota)dx$$

$$= \frac{1}{2} \{I_4(1,2,i,j,k,\iota) - j[I_3(1,j,i,k,\iota) - I_4(1,j,i,k,\iota,2)]\}$$

$$G_1(i,j,k,\iota) = \int_0^1 \sin i\pi x \; X_2(1,j) \; Z_1(k) \; Z_1(\iota)dx$$

$$= \frac{1}{2} \{I_4(2,i,j,k,\iota,1) - kI_2(1,2,i,k,j,\iota) - \iota I_2(1,2,i,\iota,j,k)$$

$$+ k\iota[I_1(1,i,k,\iota,j) - I_2(1,i,k,\iota,j,2)]\}$$

$$G_2(i,j,k,\iota) = \int_0^1 \sin i\pi x \; X_2(1,j) \; Z_1(k) \; Z_2(\iota)dx$$

$$= \frac{1}{2} \{J_3(2,i,\iota,j,k,1) - kJ_1(1,2,i,k,\iota,j) + \iota J_3(1,2,i,j,k,\iota)$$

$$- k\iota[J_2(1,i,k,j,\iota) - J_3(1,i,k,j,\iota,2)]\}$$

$$G_3(i,j,k,\iota) = \int_0^1 \sin i\pi x \; X_2(1,j) \; Z_2(k) \; Z_2(\iota)dx$$

$$= \frac{1}{2} \{I_2(2,i,k,\iota,j,1) + kI_2(1,2,i,\iota,j,k) + \iota I_2(1,2,i,k,j,\iota)$$

$$+ k\iota[I_3(1,i,j,k,\iota) - I_4(1,i,j,k,\iota,2)]\}$$

$$G_4(i,j,k,\iota) = \int_0^1 \sin i\pi x \; X_1(1,j) \; Z_1(k) \; Z_1(\iota)dx$$

$$= \frac{1}{2} \{J_3(2,i,j,k,\iota,1) - kJ_1(1,2,i,j,k,\iota) - \iota J_1(1,2,i,j,\iota,k)$$

$$+ k\iota[J_0(1,i,j,k,\iota) - J_1(1,i,j,k,\iota,2)]\}$$

$$G_5(i,j,k,\ell) = \int_0^1 \sin i\pi x \; X_1(1,j) \; Z_1(k) \; Z_2(\ell)dx$$

$$\begin{aligned}
&= \frac{1}{2} \{I_2(2,i,j,\ell,k,1) - kI_0(1,2,i,j,k,\ell) + \ell I_2(1,2,i,j,k,\ell) \\
&\quad - k\ell[I_1(1,i,j,k,\ell) - I_2(1,i,j,k,\ell,2)]\}
\end{aligned}$$

$$G_6(i,j,k,\ell) = \int_0^1 \sin i\pi x \; X_1(1,j) \; Z_2(k) \; Z_2(\ell)dx$$

$$\begin{aligned}
&= \frac{1}{2} \{J_1(2,i,j,k,\ell,1) + kJ_1(1,2,i,j,\ell,k) + \ell J_1(1,2,i,j,k,\ell) \\
&\quad + k\ell[J_2(1,i,j,k,\ell) - J_3(1,i,j,k,\ell,2)]\}
\end{aligned}$$

$$K_0(i,j,k,\ell) = \int_0^1 Z_2(i) \; Z_2(j) \; Z_2(k) \; Z_2(\ell)dx$$

$$\begin{aligned}
&= \frac{1}{8} \{3I_0(i,j,k,\ell,) + 4I_1(i,j,k,\ell,2) + I_1(i,j,k,\ell,4) \\
&\quad + i[2I_1(2,j,k,\ell,i) + I_1(4,j,k,\ell,i)] + j[2I_1(2,i,k,\ell,j) \\
&\quad + I_1(4,i,k,\ell,j)] + k[2I_1(2,i,j,\ell,k) + I_1(4,i,j,\ell,k)] \\
&\quad + \ell[2I_1(2,i,j,k,\ell) + I_1(4,i,j,k,\ell)] + ij[I_2(k,\ell,i,j) \\
&\quad - I_3(k,\ell,i,j,4)] + ik[I_2(j,\ell,i,k) - I_3(j,\ell,i,k,4)] \\
&\quad + i\ell[I_2(j,k,i,\ell) - I_3(j,k,i,\ell,4)] + jk[I_2(i,\ell,j,k) \\
&\quad - I_3(i,\ell,j,k,4)] + j\ell[I_2(i,k,j,\ell) - I_3(i,k,j,\ell,4)] \\
&\quad + k\ell[I_2(i,j,k,\ell) - I_3(i,j,k,\ell,4)] + ijk[2I_3(2,\ell,i,j,k) \\
&\quad - I_3(4,\ell,i,j,k)] + ij\ell[2I_3(2,k,i,j,\ell) - I_3(4,k,i,j,\ell)] \\
&\quad + ik\ell[2I_3(2,j,i,k,\ell) - I_3(4,j,i,k,\ell)] + jk\ell[2I_3(2,i,j,k,\ell) \\
&\quad - I_3(4,i,j,k,\ell)] + ijk\ell[3I_4(i,j,k,\ell) - 4I_5(i,j,k,\ell,2) \\
&\quad + I_5(i,j,k,\ell,4)]\}
\end{aligned}$$

$$
\begin{aligned}
K_1(i,j,k,\ell) &= \int_0^1 Z_1(i)\ Z_2(j)\ Z_2(k)\ Z_2(\ell)dx \\
&= \frac{1}{8}\{3J_1(j,k,\ell,i) + 4J_2(j,k,\ell,i,2) + J_2(j,k,\ell,i,4) \\
&\quad - i[2J_0(2,i,j,k,\ell) + J_0(4,i,j,k,\ell)] + j[2J_2(2,k,\ell,i,j) \\
&\quad + J_2(4,k,\ell,i,j)] + k[2J_2(2,j,\ell,i,k) + J_2(4,j,\ell,i,k)] \\
&\quad + \ell[2J_2(2,j,k,i,\ell) + J_2(4,j,k,i,\ell)] - ij[J_1(i,k,\ell,j) \\
&\quad - J_2(i,k,\ell,j,4)] - ik[J_1(i,j,\ell,k) - J_2(i,j,\ell,k,4)] \\
&\quad - i\ell[J_1(i,j,k,\ell) - J_2(i,j,k,\ell,4)] + jk[J_3(\ell,i,j,k) \\
&\quad - J_4(\ell,i,j,k,4)] + j\ell[J_3(k,i,j,\ell) - J_4(k,i,j,\ell,4)] \\
&\quad + k\ell[J_3(j,i,k,\ell) - J_4(j,i,k,\ell,4)] - ijk[2J_2(2,i,\ell,j,k) \\
&\quad - J_2(4,i,\ell,j,k)] - ij\ell[2J_2(2,i,k,j,\ell) - J_2(4,i,k,j,\ell)] \\
&\quad - ik\ell[2J_2(2,i,j,k,\ell) - J_2(4,i,j,k,\ell)] + jk\ell[2J_4(2,i,j,k,\ell) \\
&\quad - J_4(4,i,j,k,\ell)] - ijk\ell[3J_3(i,j,k,\ell) - 4J_4(i,j,k,\ell,2) \\
&\quad + J_4(i,j,k,\ell,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
K_2(i,j,k,\ell) &= \int_0^1 Z_1(i)\ Z_1(j)\ Z_2(k)\ Z_2(\ell)dx \\
&= \frac{1}{8}\{3I_2(k,\ell,i,j,) + 4I_3(k,\ell,i,j,2) + I_3(k,\ell,i,j,4) \\
&\quad - i[2I_1(2,i,k,\ell,j) + I_1(4,i,k,\ell,j)] - j[2I_1(2,j,k,\ell,i) \\
&\quad + I_1(4,j,k,\ell,i)] + k[2I_3(2,\ell,i,j,k) + I_3(4,\ell,i,j,k)] \\
&\quad + \ell[2I_3(2,k,i,j,\ell) + I_3(4,k,i,j,\ell)] + ij[I_0(i,j,k,\ell) \\
&\quad - I_1(i,j,k,\ell,4)] - ik[I_2(i,\ell,j,k) - I_3(i,\ell,j,k,4)] \\
&\quad - i\ell[I_2(i,k,j,\ell) - I_3(i,k,j,\ell,4)] - jk[I_2(j,\ell,i,k) \\
&\quad - I_3(j,\ell,i.k,4)] - j\ell[I_2(j,k,i,\ell) - I_3(j,k,i,\ell,4)] \\
&\quad + k\ell[I_4(i,j,k,\ell) - I_5(i,j,k,\ell,4)] + ijk[2I_1(2,i,j,\ell,k) \\
&\quad - I_1(4,i,j,\ell,k)] + ij\ell[2I_1(2,i,j,k,\ell) - I_1(4,i,j,k,\ell)] \\
&\quad - ik\ell[2I_3(2,i,k,j,\ell) - I_3(4,i,k,j,\ell)] - jk\ell[2I_3(2,j,i,k,\ell) \\
&\quad - I_3(4,j,i,k,\ell)] + ijk\ell[3I_2(i,j,k,\ell) - 4I_3(i,j,k,\ell,2) \\
&\quad + I_3(i,j,k,\ell,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
K_3(i,j,k,\iota) &= \int_0^1 Z_1(i)\; Z_1(j)\; Z_1(k)\; Z_2(\iota)dx \\
&= \frac{1}{8}\{3J_3(\iota,i,j,k) + 4J_4(\iota,i,j,k,2) + J_4(\iota,i,j,k,4) \\
&\quad - i[2J_2(2,i,\iota,j,k) + J_2(4,i,\iota,j,k)] - j[2J_2(2,j,\iota,i,k) \\
&\quad + J_2(4,j,\iota,i,k)] - k[2J_2(2,k,\iota,i,j) + J_2(4,k,\iota,i,j)] \\
&\quad + \iota[2J_4(2,i,j,k,\iota) + J_4(4,i,j,k,\iota)] + ij[J_1(i,j,\iota,k) \\
&\quad - J_2(i,j,\iota,k,4)] + ik[J_1(i,k,\iota,j) - J_2(i,k,\iota,j,4)] \\
&\quad - i\iota[J_3(i,j,k,\iota) - J_4(i,j,k,\iota,4)] + jk[J_1(j,k,\iota,i) \\
&\quad - J_2(j,k,\iota,i,4)] - j\iota[J_3(j,i,k,\iota) - J_4(j,i,k,\iota,4)] \\
&\quad - k\iota[J_3(k,i,j,\iota) - J_4(k,i,j,\iota,4)] - ijk[2J_0(2,i,j,k,\iota) \\
&\quad - J_0(4,i,j,k,\iota)] + ij\iota[2J_2(2,i,j,k,\iota) - J_2(4,i,j,k,\iota)] \\
&\quad + ik\iota[2J_2(2,i,k,j,\iota) - J_2(4,i,k,j,\iota)] + jk\iota[2J_2(2,j,k,i,\iota) \\
&\quad + J_2(4,j,k,i,\iota)] - ijk\iota[3J_1(i,j,k,\iota) - 4J_2(i,j,k,\iota,2) \\
&\quad + J_2(i,j,k,\iota,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
K_4(i,j,k,\iota) &= \int_0^1 Z_1(i)\; Z_1(j)\; Z_1(k)\; Z_1(\iota)dx \\
&= \frac{1}{8}\{3I_4(i,j,k,\iota) + 4I_5(i,j,k,\iota,2) + I_5(i,j,k,\iota,4) \\
&\quad - i[2I_3(2,i,j,k,\iota) + I_3(4,i,j,k,\iota)] - j[2I_3(2,j,i,k,\iota) \\
&\quad + I_3(4,j,i,k,\iota)] - k[2I_3(2,k,i,j,\iota) + I_3(4,k,i,j,\iota)] \\
&\quad - \iota[2I_3(2,\iota,i,j,k) + I_3(4,\iota,i,j,k)] + ij[I_2(i,j,k,\iota) \\
&\quad - I_3(i,j,k,\iota,4)] + ik[I_2(i,k,j,\iota) - I_3(i,k,j,\iota,4)] \\
&\quad + i\iota[I_2(i,\iota,j,k) - I_3(i,\iota,j,k,4)] + jk[I_2(j,k,i,\iota) \\
&\quad - I_3(j,k,i,\iota,4)] + j\iota[I_2(j,\iota,i,k) - I_3(j,\iota,i,k,4)] \\
&\quad + k\iota[I_2(k,\iota,i,j) - I_3(k,\iota,i,j,4)] - ijk[2I_1(2,i,j,k,\iota) \\
&\quad - I_1(4,i,j,k,\iota)] - ij\iota[2I_1(2,i,j,\iota,k) - I_1(4,i,j,\iota,k)] \\
&\quad - ik\iota[2I_1(2,i,k,\iota,j) - I_1(4,i,k,\iota,j)] - jk\iota[2I_1(2,j,k,\iota,i) \\
&\quad - I_1(4,j,k,\iota,i)] + ijk\iota[3I_0(i,j,k,\iota) - 4I_1(i,j,k,\iota,2) \\
&\quad + I_1(i,j,k,\iota,4)]\}
\end{aligned}
$$

$$L_0(i,j,k,\ell) = \int_0^1 X_2(1,i)\, X_2(1,j)\, X_2(1,k)\, X_2(1,\ell)dx$$

$$= \frac{1}{8}\, [3I_4(i,j,k,\ell) - 4I_5(i,j,k,\ell,2) + I_5(i,j,k,\ell,4)]$$

$$L_1(i,j,k,\ell) = \int_0^1 X_1(1,i)\, X_2(1,j)\, X_2(1,k)\, X_2(1,\ell)dx$$

$$= \frac{1}{8}\, [3J_3(i,j,k,\ell) - 4J_4(i,j,k,\ell,2) + J_4(i,j,k,\ell,4)]$$

$$L_2(i,j,k,\ell) = \int_0^1 X_1(1,i)\, X_1(1,j)\, X_2(1,k)\, X_2(1,\ell)dx$$

$$= \frac{1}{8}\, [3I_2(i,j,k,\ell) - 4I_3(i,j,k,\ell,2) + I_3(i,j,k,\ell,4)]$$

$$L_3(i,j,k,\ell) = \int_0^1 X_1(1,i)\, X_1(1,j)\, X_1(1,k)\, X_2(1,\ell)dx$$

$$= \frac{1}{8}\, [3J_1(i,j,k,\ell) - 4J_2(i,j,k,\ell,2) + J_2(i,j,k,\ell,4)]$$

$$L_4(i,j,k,\ell) = \int_0^1 X_1(1,i)\, X_1(1,j)\, X_1(1,k)\, X_1(1,\ell)dx$$

$$= \frac{1}{8}\, [3I_0(i,j,k,\ell) - 4I_1(i,j,k,\ell,2) + I_1(i,j,k,\ell,4)]$$

$$M_{00}(i,j,k,\ell) = \int_0^1 Z_2(i)\, Z_2(j)\, X_2(1,k)\, X_2(1,\ell)dx$$

$$= \frac{1}{8}\, \{I_2(i,j,k,\ell) - I_1(i,j,k,\ell,4) + i[2I_3(2,j,i,k,\ell)$$

$$- I_3(4,j,i,k,\ell)] + j[2I_3(2,i,j,k,\ell) - I_3(4,i,j,k,\ell)]$$

$$+ ij[3I_4(i,j,k,\ell) - 4I_5(i,j,k,\ell,2) + I_5(i,j,k,\ell,4)]\}$$

$$
\begin{aligned}
M_{01}(i,j,k,\ell) &= \int_0^1 Z_2(i)\ Z_2(j)\ X_1(1,k)\ X_2(1,\ell)dx \\
&= \frac{1}{8}\{J_1(i,j,k,\ell) - J_2(i,j,k,\ell,4) + i[2J_2(2,j,k,i,\ell) \\
&\quad - J_2(4,j,k,i,\ell)] + j[2J_2(2,i,k,j,\ell) - J_2(4,i,k,j,\ell)] \\
&\quad + ij[3J_3(k,i,j,\ell) - 4J_4(k,i,j,\ell,2) + J_4(k,i,j,\ell,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
M_{02}(i,j,k,\ell) &= \int_0^1 Z_2(i)\ Z_2(j)\ X_1(1,k)\ X_1(1,\ell)dx \\
&= \frac{1}{8}\{I_0(i,j,k,\ell) - I_1(i,j,k,\ell,4) + i[2I_1(2,j,k,\ell,i) \\
&\quad - I_1(4,j,k,\ell,i)] + j[2I_1(2,i,k,\ell,j) - I_1(4,i,k,\ell,j)] \\
&\quad + ij[3I_2(k,\ell,i,j) - 4I_3(k,\ell,i,j,2) + I_3(k,\ell,i,j,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
M_{10}(i,j,k,\ell) &= \int_0^1 Z_1(i)\ Z_2(j)\ X_2(1,k)\ X_2(1,\ell)dx \\
&= \frac{1}{8}\{J_3(j,i,k,\ell) - J_4(j,i,k,\ell,4) - i[2J_2(2,i,j,k,\ell) \\
&\quad - J_2(4,i,j,k,\ell)] + j[24_1(2,i,j,k,\ell) - J_4(4,i,j,k,\ell)] \\
&\quad - ij[3J_3(i,j,k,\ell) - 4J_4(i,j,k,\ell,2) + J_4(i,j,k,\ell,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
M_{11}(i,j,k,\ell) &= \int_0^1 Z_1(i)\ Z_2(j)\ X_1(1,k)\ X_2(1,\ell)dx \\
&= \frac{1}{8}\{I_2(j,k,i,\ell) - I_3(j,k,i,\ell,4) - i[2I_1(2,i,j,k,\ell) \\
&\quad - I_1(4,i,j,k,\ell)] + j[2I_3(2,k,i,j,\ell) - I_3(4,k,i,j,\ell)] \\
&\quad - ij[3I_2(i,k,j,\ell) - 4I_3(i,k,j,\ell,2) + I_3(i,k,j,\ell,4)]\}
\end{aligned}
$$

$$
\begin{aligned}
M_{12}(i,j,k,\ell) &= \int_0^1 Z_1(i)\ Z_2(j)\ X_1(1,k)\ X_1(1,\ell)dx \\
&= \frac{1}{8}\{J_1(j,k,\ell,i) - J_2(j,k,\ell,i,4) - i[2J_0(2,i,j,k,\ell) \\
&\quad - J_0(4,i,j,k,\ell)] + j[2J_2(2,k,\ell,i,j) - J_2(4,k,\ell,i,j)] \\
&\quad - ij[3J_1(i,k,\ell,j) - 4J_2(i,k,\ell,j,2) + J_2(i,k,\ell,j,4)]\}
\end{aligned}
$$

$$M_{20}(i,j,k,\iota) = \int_0^1 Z_1(i)\, Z_1(j)\, X_2(1,k)\, X_2(1,\iota)dx$$

$$= \frac{1}{8}\{I_4(i,j,k,\iota) - I_5(i,j,k,\iota,4) - i[2I_3(2,i,j,k,\iota)$$
$$- I_3(4,i,j,k,\iota)] - j[2I_3(2,j,i,k,\iota) - I_3(4,j,i,k,\iota)]$$
$$+ ij[3I_2(i,j,k,\iota) - 4I_3(i,j,k,\iota,2) + I_3(i,j,k,\iota,4)]\}$$

$$M_{21}(i,j,k,\iota) = \int_0^1 Z_1(i)\, Z_1(j)\, X_1(1,k)\, X_2(1,\iota)dx$$

$$= \frac{1}{8}\{J_3(k,i,j,\iota) - J_4(k,i,j,\iota,4) - i[2J_2(2,i,k,j,\iota)$$
$$- J_2(4,i,k,j,\iota)] - j[2J_2(2,j,k,i,\iota) - J_2(4,j,k,i,\iota)]$$
$$+ ij[3J_1(i,j,k,\iota) - 4J_2(i,j,k,\iota,2) + J_2(i,j,k,\iota,4)]\}$$

$$M_{22}(i,j,k,\iota) = \int_0^1 Z_1(i)\, Z_1(j)\, X_1(1,k)\, X_1(1,\iota)dx$$

$$= \frac{1}{8}\{I_2(k,\iota,i,j) - I_3(k,\iota,i,j,4) - i[2I_1(2,i,k,\iota,j)$$
$$- I_1(4,i,k,\iota,j)] - j[2I_1(2,j,k,\iota,i) - I_1(4,j,k,\iota,i)]$$
$$+ ij[3I_0(i,j,k,\iota) - 4I_1(i,j,k,\iota,2) + I_1(i,j,k,\iota,4)]\}$$

$$T_1(i,j,k) = \int_0^1 Z_1(i)\, X_1(1,j)\, X_1(1,k)dx$$

$$= I_2(1,1,j,k,i,1) - i\, I_0(1,1,1,i,j,k)$$

$$T_2(i,j,k) = \int_0^1 Z_2(i)\, X_1(1,j)\, X_1(1,k)dx$$

$$= J_1(1,1,i,j,k,1) + i\, J_1(1,1,1,j,k,i)$$

$$T_9(i,j,k) = \int_0^1 Z_1(i)\, X_2(j,1)\, X_2(k,1)dx$$

$$= I_4(j,k,i,1,1,1) - i\, J_2(1,i,j,k,1,1)$$

$$T_{10}(i,j,k) = \int_0^1 Z_2(i)\ X_2(j,1)\ X_2(k,1)dx$$

$$= J_3(i,j,k,1,1,1) + i\ J_3(1,j,k,i,1,1)$$

$$T_{14}(i,j,k) = \int_0^1 Z_2(i)\ X_2(1,j)\ X_3(1,k)dx$$

$$= I_4(1,i,j,k,1,1) + i\ I_4(1,1,i,j,k,1)$$

$$T_{16}(i,j,k) = \int_0^1 Z_1(i)\ X_2(1,j)\ X_3(1,k)dx$$

$$= J_5(1,i,j,k,1,1) - i\ J_3(1,1,i,j,k,1)$$

$$Q_1(i,j,k) = \int_0^1 Z_3(i)\ Z_2(j)\ X_1(1,k)dx$$

$$= C_1(1,i,k,j) + i[C_1(i,1,k,j) - I_1(1,i,j,k,1)] - ijI_1(1,1,i,k,j)$$

where $Z_3(i) = X_2(1,i) + i[X_2(i,1) - \sin i\pi x]$

$$Q_2(i,j,k) = \int_0^1 Z_3(i)\ Z_2(j)\ X_2(1,k)dx$$

$$= C_3(1,i,k,j) + i[C_3(i,1,k,j) - J_2(1,i,j,k,1)] - ijJ_2(1,1,i,j,k)$$

$$Q_3(i,j,k) = \int_0^1 Z_3(i)\ Z_1(j)\ X_1(1,k)dx$$

$$= C_2(1,i,k,j) + i[C_2(i,1,k,j) - J_2(1,i,k,j,1)] + ijJ_0(1,1,i,j,k)$$

$$Q_4(i,j,k) = \int_0^1 Z_3(i)\ Z_1(j)\ X_2(1,k)dx$$

$$= C_4(1,k,i,j) + i[C_4(i,1,k,j) - I_3(1,i,j,k,1)] + ijI_1(1,1,i,j,k)$$

$$Q_5(i,j,k) = \int_0^1 Z_4(i)\ Z_2(j)\ X_1(1,k)dx$$

$$= T_2(j,i,k) - i[C_3(k,1,i,j) - J_2(1,j,k,i,1)] + ijJ_2(1,1,k,i,j)$$

where $Z_4(i) = X_1(i,1) - i[X_3(1,i) - \cos i\pi x]$

$$Q_6(i,j,k) = \int_0^1 Z_4(i)\ Z_2(j)\ X_2(1,k)dx$$

$$= C_1(1,k,i,j) - i[T_{14}(j,k,i) - I_3(1,j,i,k,1)] + ijI_3(1,1,i,j,k)$$

$$Q_7(i,j,k) = \int_0^1 Z_4(i)\ Z_1(j)\ X_1(1,k)dx$$

$$= T_1(j,i,k) - i[C_4(k,1,i,j) - I_3(1,k,i,j,1)] - ijI_1(1,1,j,k,i)$$

$$Q_8(i,j,k) = \int_0^1 Z_4(i)\ Z_1(j)\ X_2(1,k)dx$$

$$= C_2(1,k,i,j) - i[T_{16}(j,k,i) - J_4(1,i,j,k,1)] - ijJ_2(1,1,j,i,k)$$

$$U_1(i,j,k) = \int_0^1 \cos i\pi x\ Z_1(j)\ Z_1(k)dx$$

$$= I_5(i,j,k,1,1) - jI_3(1,j,i,k,1) - kI_3(1,k,i,j,1) + jkI_1(1,1,j,k,i)$$

$$U_2(i,j,k) = \int_0^1 \sin i\pi x\ Z_1(j)\ Z_1(k)dx$$

$$= J_4(i,j,k,1,1) - jJ_2(1,i,j,k,1) - kJ_2(1,i,k,j,1) + jkJ_0(1,1,i,j,k)$$

$$U_3(i,j,k) = \int_0^1 \cos i\pi x\ Z_1(j)\ Z_2(k)dx$$

$$= J_4(k,i,j,1,1) - jJ_2(1,j,k,i,1) + kJ_4(1,i,j,k,1) - jkJ_2(1,1,j,i,k)$$

$$U_4(i,j,k) = \int_0^1 \sin i\pi x \; Z_1(j) \; Z_2(k)dx$$

$$= I_3(i,k,j,1,1) - jI_1(1,i,j,k,1) + kI_3(1,i,j,k,1) - jkI_1(1,1,i,j,k)$$

$$U_5(i,j,k) = \int_0^1 \cos i\pi x \; Z_2(j) \; Z_2(k)dx$$

$$= I_3(j,k,i,1,1) + jI_3(1,k,i,j,1) + kI_3(1,j,i,k,1) + jkI_3(1,1,i,j,k)$$

$$U_6(i,j,k) = \int_0^1 \sin i\pi x \; Z_2(j) \; Z_2(k)dx$$

$$= J_2(i,j,k,1,1) + jJ_2(1,i,k,j,1) + kJ_2(1,i,j,k,1) + jkJ_2(1,1,i,j,k)$$

$$V_4(i,j,k) = \int_0^1 X_2(1,i) \; Z_1(j) \; Z_1(k)dx$$

$$= J_5(1,i,j,k,1,1) - jJ_3(1,1,j,i,k,1) - kJ_3(1,1,k,i,j,1) + jkJ_1(1,1,1,j,k,i)$$

$$V_5(i,j,k) = \int_0^1 X_2(1,i) \; Z_1(j) \; Z_2(k)dx$$

$$= I_4(1,k,i,j,1,1) - jI_2(1,1,j,k,i,1) + kI_4(1,1,i,j,k,1) - jkI_2(1,1,1,j,i,k)$$

$$V_6(i,j,k) = \int_0^1 X_2(1,i) \; Z_2(j) \; Z_2(k)dx$$

$$= J_3(1,j,k,i,1,1) + jJ_3(1,1,k,i,j,1) + kJ_3(1,1,j,i,k,1) + jkJ_3(1,1,1,i,j,k)$$

$$V_7(i,j,k) = \int_0^1 X_2(i,1) \; Z_2(j) \; Z_2(k)dx$$

$$= J_3(i,j,k,1,1,1) + jJ_3(1,i,k,j,1,1) + kJ_3(1,i,j,k,1,1) + jkJ_3(1,1,i,j,k,1)$$

$$V_8(i,j,k) = \int_0^1 X_2(i,1)\, Z_1(j)\, Z_2(k)dx$$

$$= I_4(i,k,j,1,1,1) - jI_2(1,i,j,k,1,1) + kI_4(1,i,j,k,1,1) - jkI_2(1,1,i,j,k,1)$$

$$V_9(i,j,k) = \int_0^1 X_2(i,1)\, Z_1(j)\, Z_1(k)dx$$

$$= J_5(i,j,k,1,1,1) - jJ_3(1,i,j,k,1,1) - kJ_3(1,i,k,j,1,1) + jkJ_1(1,1,i,j,k,1)$$

$$W_1(i,j,k) = \int_0^1 Z_1(i)\, Z_1(j)\, Z_1(k)dx$$

$$= I_6(i,j,k,1,1,1) - iI_4(1,i,j,k,1,1) - jI_4(1,j,i,k,1,1)$$

$$- kI_4(1,k,i,j,1,1) + ijI_2(1,1,i,j,k,1) + ikI_2(1,1,i,k,j,1)$$

$$+ jkI_2(1,1,j,k,i,1) - ijkI_0(1,1,1,i,j,k)$$

$$W_2(i,j,k) = \int_0^1 Z_1(i)\, Z_2(j)\, Z_2(k)dx$$

$$= I_4(j,k,i,1,1,1) - iI_2(1,i,j,k,1,1) + jI_4(1,k,i,j,1,1)$$

$$+ kI_4(1,j,i,k,1,1) - ijI_2(1,1,i,k,j,1) - ikI_2(1,1,i,j,k,1)$$

$$+ jkI_4(1,1,i,j,k,1) - ijkI_2(1,1,1,i,j,k)$$

$$W_3(i,j,k) = \int_0^1 Z_1(i)\, Z_1(j)\, Z_2(k)dx$$

$$= J_5(k,i,j,1,1,1) - iJ_3(1,i,k,j,1,1) - jJ_3(1,j,k,i,1,1)$$

$$+ kJ_5(1,i,j,k,1,1) + ijJ_1(1,1,i,j,k,1) - ikJ_3(1,1,i,j,k,1)$$

$$- jkJ_3(1,1,j,i,k,1) + ijkJ_1(1,1,1,i,j,k)$$

$$W_4(i,j,k) = \int_0^1 Z_2(i)\ Z_2(j)\ Z_2(k)dx$$

$$= J_3(i,j,k,1,1,1) + iJ_3(1,j,k,i,1,1) + jJ_3(1,i,k,j,1,1)$$

$$+ kJ_3(1,i,j,k,1,1) + ijJ_3(1,1,k,i,j,1) + ikJ_3(1,1,j,i,k,1)$$

$$+ jkJ_3(1,1,i,j,k,1) + ijkJ_3(1,1,1,i,j,k)$$

$$P_1(i,j,k) = \int_0^1 [\sin i\pi x - X_2(i,1)]\ Z_2(j)\ Z_2(k)dx$$

$$= U_6(i,j,k) - V_7(i,j,k)$$

$$P_2(i,j,k) = \int_0^1 [\sin i\pi x - X_2(i,1)]\ Z_1(j)\ Z_2(k)dx$$

$$= U_4(i,j,k) - V_8(i,j,k)$$

$$P_3(i,j,k) = \int_0^1 [\sin i\pi x - X_2(i,1)]\ Z_1(j)\ Z_1(k)dx$$

$$= U_2(i,j,k) - V_9(i,j,k)$$

$$P_4(i,j,k) = \int_0^1 [\cos i\pi x - X_3(i,1)]\ Z_2(j)\ Z_2(k)dx$$

$$= U_5(i,j,k) - B_1(1,i,j,k)$$

$$P_5(i,j,k) = \int_0^1 [\cos i\pi x - X_3(i,1)]\ Z_1(j)\ Z_2(k)dx$$

$$= U_3(i,j,k) - B_2(1,i,j,k)$$

$$P_6(i,j,k) = \int_0^1 [\cos i\pi x - X_3(i,1)]\ Z_1(j)\ Z_1(k)dx$$

$$= U_1(i,j,k) - B_3(1,i,j,k)$$

The energy integrals are evaluated in closed form and they are expressed in terms of the integrals defined above. These integrals are given below.

$$F_5^0 = \int_0^1 \int_0^1 f_5(x,y)dxdy = J(1)\ J(1) = \frac{4}{\pi^2}$$

$$G_{55}^{11} = \int_0^1 \int_0^1 f_{5,1}(x,y)f_{5,1}(x,y)dxdy = \pi^2 I_2(1,1)I_0(1,1) = \frac{\pi^2}{4}$$

$$G_{55}^{12} = \int_0^1 \int_0^1 f_{5,1}(x,y)f_{5,2}(x,y)dxdy = \pi^2 J_1(1,1)J_1(1,1) = 0$$

$$G_{55}^{22} = \int_0^1 \int_0^1 f_{5,2}(x,y)f_{5,2}(x,y)dxdy = \pi^2 I_0(1,1)I_2(1,1) = \frac{\pi^2}{4}$$

$$J_{1nm} = \int_0^1 f_1(1,y;n,m)dy = \sin\pi\ [\cos n\pi\ I(m) + \sin n\pi\ J(m)] = 0$$

$$J_{2nm} = \int_0^1 f_2(1,y;n,m)dy = \cos n\pi\ (1-\cos\pi)I(m)$$
$$+ \sin n\pi\ (1-\sin\pi)J(m) = 0 \qquad m \neq 0$$

$$J_{1nm}^1 = \int_0^1 f_{1,1}(X,0;n,m)dx = \pi\{[\ I_2(1,n) - nI_0(1,n)]\cos(0)$$
$$+ [J_1(n,1) + nJ_1(1,n)]\sin(0)\} = 0$$

$$K_{11nmpq}^{11} = \int_0^1 f_{1,1}(X,0;n,m)f_{1,1}(X,0;p,q)dx$$
$$= \pi^2[I_4(n,p,1,1) - nI_2(1,n,p,1) - pI_2(1,p,n,1)$$
$$+ npI_0(1,1,n,p)]$$

$$K_{22nmpq}^{22} = \int_0^1 f_{2,2}(1,y;n,m)f_{2,2}(1,y;p,q)dy = 4mq\pi^2(-1)^{n+p}I_0(m,q)$$

$$K^{33}_{22nmpq} = \int_0^1 f_{2,3}(X,0;n,m)f_{2,3}(X,0;p,q)dx$$

$$= \pi^4[(1+n^2)(1+p^2)\ I_4(n,p,1,1) - (n^2+p^2+2n^2p^2)I_3(n,p,1)$$

$$+n^2p^2I_2(n,p) - 2n(1+p^2)I_2(1,n,p,1) + 2np^2I_1(1,n,p)$$

$$- 2p(1+n^2)I_2(1,p,n,1) + 2n^2pI_1(1,p,n) + 4npI_0(1,1,n,p)]$$

$$K^{44}_{11nmpq} = \int_0^1 f_{1,4}(1,y;n,m)f_{1,4}(1,y;p,q)dy = 0$$

$$F^0_{3nm} = \int_0^1 \int_0^1 f_3(x,y;n,m)dxdy = J(n)\ J(m)$$

$$F^0_{4nm} = \int_0^1 \int_0^1 f_4(x,y;n,m)dxdy = I_0(1,n)J_1(1,m) - J_1(1,n)I_0(1,m)$$

$$F^1_{1nm} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m)dxdy = \pi\{[I_2(n,1) - nI_0(1,n)]\ I(m)$$

$$+ [J_1(n,1) + nJ_1(1,n)]\ J(m)\}$$

$$= \pi[J_1(n,1) + nJ_1(1,n)]\ J(m) \qquad \text{for } m \neq 0$$

$$F^2_{1nm} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m)dxdy = m\pi[J_1(1,n)I(m) - nJ_1(1,n)J(m)]$$

$$= - m\pi J_1(1,n)J(m) \qquad \text{for } m \neq 0$$

$$F^1_{2nm} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m)dxdy = \pi\{[nJ_1(n,1) + J_1(1,n)-nJ(n)]\ I(m)$$

$$+ [nI(n) - nI_2(n,1) + I_0(1,n)]\ J(m)\}$$

$$= 0 \qquad \text{for } n,m \neq 0$$

$$F^2_{2nm} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)dxdy = m\pi\{[J(n) - J_1(n,1)]\ I(m)$$

$$- [I(n) - I_2(1,n)]\ J(m)\}$$

$$= m\pi I_2(1,n)\ J(m) \qquad \text{for } n,m \neq 0$$

$$G^{53}_{45nm} = \int_0^1 \int_0^1 f_{4,5}(x,y;n,m) f_{5,3}(x,y)dxdy$$

$$= -\pi^4\{[I(1,n,1) + nI_1(1,1,n)][J_2(1,m,1) - m\, J_0(1,1,m)]$$

$$- [J_2(1,n,1) - n\, J_0(1,1,n)][I_1(1,m,1) + mI_1(1,1,m)]\}$$

$$G^{54}_{45nm} = \int_0^1 \int_0^1 f_{4,5}(x,y;n,m)\ f_{5,4}(x,y)dxdy = G^{53}_{45nm}$$

$$G^{55}_{45nm} = \int_0^1 \int_0^1 f_{4,5}(x,y;n,m)\ f_{5,5}(x,y)dxdy$$

$$= -\pi^4\{[J_2(n,1,1) + nJ_2(1,n,1)][I_3(m,1,1) - mI_1(1,m,1)]$$

$$- [I_3(n,1,1) - nI_1(1,n,1)][J_2(m,1,1) + mJ_2(1,m,1)]\}$$

$$G^{11}_{35nm} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m)\ f_{5,1}(x,y)dxdy = n\pi^2\ I_2(n,1)I_0(1,m)$$

$$= \begin{cases} \pi^2/4 & n = m = 1 \\ 0 & \text{all other } n,m \end{cases}$$

$$G^{12}_{35nm} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m)\ f_{5,2}(x,y)dxdy = n\pi^2\ J_1(1,n)J_1(m,1)$$

$$G^{21}_{35nm} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m)\ f_{5,1}(x,y)dxdy = G^{12}_{35mn}$$

$$G^{22}_{35nm} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m)\ f_{5,2}(x,y)dxdy = G^{11}_{35mn}$$

$$G_{35nm}^{33} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m)\ f_{5,3}(x,y)dxdy = n^2\pi^4\ I_0(1,n)I_0(1,m)$$

$$= \begin{cases} \pi^4/4 & n = m = 1 \\ 0 & \text{all other } n,m \end{cases}$$

$$G_{35nm}^{34} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m)\ f_{5,4}(x,y)dxdy = G_{35nm}^{33}$$

$$G_{35nm}^{35} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m)\ f_{5,5}(x,y)dxdy = -\ I^2\pi^4 J_1(n,1)J_1(m,1)$$

$$G_{35nm}^{43} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m)\ f_{5,3}(x,y)dxdy = G_{35mn}^{33}$$

$$G_{35nm}^{44} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m)\ f_{5,4}(x,y)dxdy = G_{35mn}^{33}$$

$$G_{35nm}^{45} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m)\ f_{5,5}(x,y)dxdy = G_{35mn}^{35}$$

$$G_{35nm}^{53} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m)\ f_{5,3}(x,y)dxdy = -\ nm\pi^4 J_1(1,n)J_1(1,m)$$

$$G_{35nm}^{54} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m)\ f_{5,4}(x,y)dxdy = G_{35nm}^{53}$$

$$G_{35nm}^{55} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m)\ f_{5,5}(x,y)dxdy = nm\pi^4 I_2(n,1)I_2(m,1)$$

$$G^{11}_{35nm} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m)\ f_{5,1}(x,y)dxdy$$

$$= \pi^2\{[J_2(n,1,1) + nJ_2(1,n,1)]\ I_1(1,1,m)$$

$$- [I_3(n,1,1) - nI_1(1,n,1)]\ J_0(1,1,m)\}$$

$$G^{12}_{45nm} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m)\ f_{5,2}(x,y)dxdy$$

$$= \pi^2\{[I_1(1,n,1) + nI_1(1,1,n)]\ J_2(1,m,1)$$

$$- [J_2(1,n,1) - nJ_0(1,1,n)]\ I_1(1,m,1)\}$$

$$G^{21}_{45nm} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m)\ f_{5,1}(x,y)dxdy = -\ G^{12}_{45mn}$$

$$G^{22}_{45nm} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m)\ f_{5,2}(x,y)dxdy = -\ G^{11}_{45mn}$$

$$G^{33}_{45nm} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m)\ f_{5,3}(x,y)dxdy$$

$$= -\ \pi^4\{[2nJ_2(1,n,1) - (1+n^2)J_0(1,1,n)]\ I_1(1,1,m)$$

$$+ [2nJ_1(1,n,1) + (1+n^2)J_1(1,1,n)]\ J_0(1,1,m)\}$$

$$G^{34}_{45nm} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m)\ f_{5,4}(x,y)dxdy = G^{33}_{45nm}$$

$$G^{35}_{45nm} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m)\ f_{5,5}(x,y)dxdy$$

$$= \pi^4\{[2nI_3(n,1,1) - (1+n^2)I_1(1,n,1)]\ J_2(1,m,1)$$

$$+ [2nJ_2(n,1,1) + (1+n^2)J_2(1,n,1)]\ I_1(1,m,1)\}$$

$$G_{45nm}^{43} = \int_0^1 \int_0^1 f_{4,4}(x,y;n,m)\ f_{5,3}(x,y)dxdy = -\ G_{45mn}^{33}$$

$$G_{45nm}^{44} = \int_0^1 \int_0^1 f_{4,4}(x,y;n,m)\ f_{5,4}(x,y)dxdy = -\ G_{45mn}^{33}$$

$$G_{45nm}^{45} = \int_0^1 \int_0^1 f_{4,4}(x,y;n,m)\ f_{5,5}(x,y)dxdy = -\ G_{45mn}^{33}$$

$$G_{35nm}^{00} = \int_0^1 \int_0^1 f_3(x,y;n,m)\ f_5(x,y)dxdy = I_0(n,1)I_0(m,1)$$

$$= \begin{cases} 1/4 & n = m = 1 \\ 0 & \text{all other } n,m \end{cases}$$

$$G_{45nm}^{00} = \int_0^1 \int_0^1 f_4(x,y;n,m)\ f_5(x,y)dxdy$$

$$= J_0(1,1,n)I_1(1,1,m) - I_1(1,1,n)J_0(1,1,m)$$

$$G_{15nm}^{10} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m)\ f_5(x,y)dxdy$$

$$= \pi\{[J_2(1,n,1) - nJ_0(1,1,n)]J_1(1,m)$$
$$+ [I_1(1,n,1) + nI_1(1,1,n)]I_0(1,m)\}$$

$$G_{25nm}^{20} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)f_5(x,y)dxdy$$

$$= m\pi\{[I_0(1,n) - I_1(1,n,1)]J_1(1,m)$$
$$- [J_1(1,n) - J_2(1,n,1)]I_0(1,m)\}$$

$$H^{101}_{355nm} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_5(x,y) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^2 I_2(1,n,1)\ I_0(1,1,m)$$

$$H^{202}_{355nm} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_5(x,y) f_{5,2}(x,y)\,dxdy = H^{101}_{355mn}$$

$$H^{110}_{455nm} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{5,1}(x,y) f_5(x,y)\,dxdy$$

$$= \pi^2\{[I_2(1,n,1,1) + nI_2(1,1,n,1)]\ J_1(1,1,1,m)$$

$$- [J_3(1,n,1,1) - nJ_1(1,1,n,1)] I_0(1,1,1,m)\}$$

$$H^{202}_{455nm} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_5(x,y) f_{5,2}(x,y)\,dxdy = - H^{110}_{455mn}$$

$$H^{011}_{355nm} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= \pi^2 J_2(n,1,1)\ I_0(1,1,m)$$

$$H^{022}_{355nm} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy = H^{011}_{355mn}$$

$$H^{011}_{455nm} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= \pi^2[I_2(1,n,1,1)\ J_1(1,1,1,m) - J_3(1,n,1,1)\ I_0(1,1,1,m)]$$

$$H^{022}_{455nm} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy = - H^{011}_{455mn}$$

$$H^{111}_{155nm} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= \pi^3\{[I_4(n,1,1,1) - nI_2(1,n,1,1)]\ I_1(1,1,m)$$

$$+ [J_3(n,1,1,1) + nJ_3(1,n,1,1)]\ J_0(1,1,m)\}$$

$$H^{122}_{155nm} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= \pi^3\{[I_2(1,1,n,1) - nI_0(1,1,1,n)]\ J_3(m,1,1)$$

$$+ [J_1(1,1,n,1) + nJ_1(1,1,1,n)]\ I_2(m,1,1)\}$$

$$H^{212}_{155nm} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= m\pi^3[J_1(1,1,n,1)\ J_2(1,m,1) - I_2(1,1,n,1)\ I_1(1,m,1)]$$

$$H^{112}_{255nm} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= \pi^3\{[I_2(1,1,n,1) + n(I_2(1,n,1,1) - I_1(1,n,1))]\ J_2(1,m,1)$$

$$+ [J_1(1,1,n,1) - n(J_3(1,n,1,1) - J_2(1,n,1))]\ I_1(1,m,1)\}$$

$$H^{211}_{255nm} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= m\pi^3\{[J_2(n,1,1) - J_3(n,1,1,1)]\ I_1(1,1,m)$$

$$- [I_3(n,1,1) - I_4(n,1,1,1)]\ J_0(1,1,m)\}$$

$$H^{222}_{255nm} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= m\pi^3\{[J_0(1,1,n) - J_1(1,1,n,1)]\ I_3(m,1,1)$$

$$- [I_1(1,1,n) - I_2(1,1,n,1)]\ J_2(m,1,1)\}$$

$$I^{1111}_{3555nm} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^4 J_4(n,1,1,1)\ I_0(1,1,1,m)$$

$$I^{1122}_{3555nm} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{5,1}(x,y) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= n\pi^4 I_2(1,1,n,1)\ I_2(1,m,1,1)$$

$$I^{2112}_{3555nm} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy = I^{1122}_{3555mn}$$

$$I^{2222}_{3555nm} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy = I^{1111}_{3555mn}$$

$$I^{1111}_{4555nm} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= \pi^4 \{ [J_4(n,1,1,1,1) + nJ_4(1,n,1,1,1)]\ I_1(1,1,1,1,m)$$

$$- [I_5(n,1,1,1,1) - nI_3(1,n,1,1,1)]\ J_0(1,1,1,1,m) \}$$

$$I^{1122}_{4555nm} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{5,1}(x,y) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= \pi^4 \{ [J_2(1,1,n,1,1) + nJ_2(1,1,1,n,1)]\ I_3(1,1,m,1,1)$$

$$- [I_3(1,1,n,1,1) - nI_1(1,1,1,n,1)]\ J_2(1,1,m,1,1) \}$$

$$I^{2112}_{4555nm} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{5,1}(x,y) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy = -\ I^{1122}_{4555mn}$$

$$I^{2222}_{4555nm} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{5,2}(x,y) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy = -\ I^{1111}_{4555mn}$$

$$G^{00}_{33nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_3(x,y;p,q)\,dxdy$$

$$= I_0(n,p)\ I_0(m,q) = \begin{cases} 1/4 & \text{for} \quad n = p \text{ and } m = q \\ 0 & \text{for} \quad n \neq p \text{ or } m \neq q \end{cases}$$

$$G^{11}_{33nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q)\,dxdy$$

$$= np\pi^2 I_2(n,p)\ I_0(m,p) = np\pi^2\ G^{00}_{33nmpq}$$

$$G^{22}_{33nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q)\,dxdy$$

$$= mq\pi^2 I_0(n,p)\ I_2(m,q) = mq\pi^2\ G^{00}_{33mnqp}$$

$$G^{33}_{33nmpq} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{3,3}(x,y;p,q)\,dxdy = n^2p^2\pi^4\ G^{00}_{33nmpq}$$

$$G^{34}_{33nmpq} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{3,4}(x,y;p,q)\,dxdy = n^2p^2\pi^4\ G^{00}_{33nmpq}$$

$$G^{44}_{33nmpq} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m) f_{3,4}(x,y;p,q)\,dxdy = m^2q^2\pi^4\ G^{00}_{33nmpq}$$

$$G^{55}_{33nmpq} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m) f_{3,5}(x,y;p,q)\,dxdy = nmpq\pi^4\ G^{00}_{33nmpq}$$

$$G^{12}_{33nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q)\,dxdy$$

$$= nq\pi^2\ J_1(p,n)\ J_1(m,q) = nq\pi^2\ F12_{nmpq}$$

where $F12_{nmpq} = J_1(p,n)\ J_1(m,q)$

$$G_{33nmpq}^{35} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{3,5}(x,y;p,q)\,dxdy = -\, n^2 pq\pi^4 \, F12pmnq$$

$$G_{33nmpq}^{45} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m) f_{3,5}(x,y;p,q)\,dxdy = -\, m^2 pq\pi^4 \, F12pmnq$$

$$G_{34nmpq}^{11} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q)\,dxdy$$

$$= n\pi^2\{[J_2(p,n,1) + pJ_2(1,n,p)]\, I_1(1,m,q)$$

$$- [I_3(n,p,1) - pI_1(1,p,n)]\, J_0(1,m,q)\}$$

$$G_{34nmpq}^{22} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q)\,dxdy = -\, G_{34mnqp}^{11}$$

$$G_{34nmpq}^{12} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,2}(x,y;p,q)\,dxdy$$

$$= n\pi^2\{I_1(1,p,n)[J_2(m,q,1) - qJ_0(1,m,q)]$$

$$- J_2(1,n,p)[I_1(m,q,1) + qI_1(1,m,q)]\}$$

$$G_{34nmpq}^{21} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,1}(x,y;p,q)\,dxdy = -\, G_{34mnqp}^{21}$$

$$G_{34nmpq}^{33} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{4,3}(x,y;p,q)\,dxdy$$

$$= -\, n^2\pi^4\{[2pJ_2(n,p,1) - (1+p^2)J_0(1,n,p)]\, I_1(1,m,q)$$

$$+ [2pI_1(n,p,1) + (1+p^2)I_1(1,n,p)]\, J_0(1,m,q)\} = -\, n^2 F38_{nmpq}$$

$$G_{34nmpq}^{34} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{4,4}(x,y;p,q)\,dxdy = n^2 F38_{mnqp}$$

$$G^{43}_{34nmpq} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m) f_{4,3}(x,y;p,q)\,dxdy = - m^2 F38_{nmpq}$$

$$G^{44}_{34nmpq} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m) f_{4,4}(x,y;p,q)\,dxdy = m^2 F38_{mnqp}$$

$$G^{35}_{34nmpq} = \int_0^1 \int_0^1 f_{3,3}(x,y;n,m) f_{4,5}(x,y;p,q)\,dxdy$$

$$= - n^2\pi^4\{[I_1(n,p,1) + pI_1(1,n,p)][J_2(m,q,1) - qJ_0(1,m,q)]$$

$$- [J_2(n,p,1) - pJ_0(1,n,p)][I_1(m,q,1) + qI_1(1,m,q)]\}$$

$$= - n^2 F30_{nmpq}$$

$$G^{45}_{34nmpq} = \int_0^1 \int_0^1 f_{3,4}(x,y;n,m) f_{4,5}(x,y;pq)\,dxdy = - m^2 F30_{nmpq}$$

$$G^{55}_{34nmpq} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m) f_{4,5}(x,y;p,q)\,dxdy$$

$$= nm\pi^4\{[J_2(p,n,1) + pJ_2(1,n,p)][I_3(m,q,1) - qI_1(1,m,q)]$$

$$- [I_3(n,p,1) - pI_1(1,p,n)][J_2(q,m,1) + qJ_2(1,m,q)]\}$$

$$G^{53}_{34nmpq} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m) f_{4,3}(x,y;p,q)\,dxdy$$

$$= nm\pi^4\{[2pI_3(n,p,1) - (1+p^2)I_1(1,p,n)]J_2(1,m,q)$$

$$+ [2pJ_2(p,n,1) + (1+p^2)J_2(1,n,p)]I_1(1,q,m)\}$$

$$= nmF58_{nmpq}$$

$$G^{54}_{34nmpq} = \int_0^1 \int_0^1 f_{3,5}(x,y;n,m) f_{4,4}(x,y;p,q)\,dxdy = - nmF58_{mnqp}$$

$$G_{44nmpq}^{11} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q)\,dxdy$$

$$\begin{aligned}
= \pi^2\{ & [I_2(n,p,1,1) + nI_2(1,p,n,1) + pI_2(1,n,p,1) \\
& \quad + npI_2(1,1,n,p)]\ I_2(1,1,m,q) \\
& - [J_3(n,p,1,1) + nJ_3(1,n,p,1) - pJ_1(1,n,p,1) \\
& \quad - npJ_1(1,1,p,n)]\ J_1(1,1,q,m) \\
& - [J_3(p,n,1,1) - nJ_1(1,n,p,1) + pJ_3(1,n,p,1) \\
& \quad - npJ_1(1,1,n,p)]\ J_1(1,1,m,q) \\
& + [I_4(n,p,1,1) - nI_2(1,n,p,1) - pI_2(1,p,n,1) \\
& \quad + npI_0(1,1,n,p)]\ I_0(1,1,m,q)\}
\end{aligned}$$

$$G_{44nmpq}^{22} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q)\,dxdy = G_{44mnqp}^{11}$$

$$G_{44nmpq}^{12} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,2}(x,y;p,q)\,dxdy$$

$$\begin{aligned}
= \pi^2\{ & [J_1(1,n,p,1) + nJ_1(1,1,p,n)][J_3(1,m,q,1) - qJ_1(1,1,q,m)] \\
& - [I_2(1,n,p,1) + nI_2(1,1,n,p)][I_2(1,q,m,1) + qI_2(1,1,m,q)] \\
& - [I_2(1,p,n,1) - nI_0(1,1,n,p)][I_2(1,m,q,1) - qI_0(1,1,m,q)] \\
& + [J_3(1,n,p,1) - nJ_1(1,1,n,p)][J_1(1,m,q,1) + qJ_1(1,1,m,q)]\}
\end{aligned}$$

$$G^{35}_{44nmpq} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m) f_{4,5}(x,y;p,q)\,dxdy$$

$$= \pi^4\{[2nJ_3(p,n,1,1) + 2npJ_3(1,n,p,1) + (1+n^2)(J_1(1,n,p,1)$$
$$+ pJ_1(1,1,n,p)][J_3(1,m,q,1) - qJ_1(1,1,q,m)]$$
$$- [2n(I_4(n,p,1,1) - pI_2(1,p,n,1)) - (1+n^2)(I_2(1,n,p,1)$$
$$- pI_0(1,1,n,p))][I_2(1,q,m,1) + qI_2(1,1,m,q)]$$
$$+ [2n(I_2(n,p,1,1) + pI_2(1,n,p,1)) + (1+n^2)(I_2(1,p,n,1)$$
$$+ pI_2(1,1,n,p))][I_2(1,m,q,1) - qI_0(1,1,m,q)]$$
$$- [2n(J_3(n,p,1,1) - pJ_1(1,n,p,1)) + (1+n^2)(J_3(1,n,p,1)$$
$$- pJ_1(1,1,p,n))][J_1(1,m,q,1) + qJ_1(1,1,m,q)]\}$$

$$G^{45}_{44nmpq} = \int_0^1 \int_0^1 f_{4,4}(x,y;n,m) f_{4,5}(x,y;p,q)\,dxdy = G^{35}_{44mnqp}$$

$$G^{33}_{44nmpq} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m) f_{4,3}(x,y;p,q)\,dxdy$$

$$= \pi^4\{[4npI_4(n,p,1,1) - 2n(1+p^2)I_2(1,p,n,1) + 2p(1+n^2)I_2(1,n,p,1)$$
$$+ (1+n^2)(1+p^2)I_0(1,1,n,p)]\ I_2(1,1,m,q)$$
$$+ [4npJ_3(p,n,1,1) + 2n(1+p^2)\ J_3(1,n,p,1) - 2p(1+n^2)\ J_1(1,n,p,1)$$
$$- (1+n^2)(1+p^2)\ J_1(1,1,n,p)]\ J_1(1,1,q,m)$$
$$+ [4npJ_3(n,p,1,1) - 2n(1+p^2)\ J_1(1,n,p,1) + 2p(1+n^2)\ J_3(1,n,p,1)$$
$$- (1+n^2)(1+p^2)\ J_1(1,1,p,n)]\ J_1(1,1,m,q)$$
$$+ [4npI_2(n,p,1,1) + 2n(1+p^2)\ I_2(1,n,p,1) + 2p(1+n^2)\ I_2(1,p,n,1)$$
$$+ (1+n^2)(1+p^2)\ I_2(1,1,n,p)]\ I_0(1,1,m,q)\}$$

$$G^{44}_{44nmpq} = \int_0^1 \int_0^1 f_{4,4}(x,y;n,m) f_{4,4}(x,y;p,q)\,dxdy = G^{33}_{44mnqp}$$

$$G^{34}_{44nmpq} = \int_0^1 \int_0^1 f_{4,3}(x,y;n,m) f_{4,4}(x,y;p,q)\,dxdy$$

$$= -\pi^4\{[2nI_2(1,p,n,1) - (1+n^2)I_0(1,1,n,p)][(1+q^2)\ I_2(1,1,m,q) + 2qI_2(1,q,m,1)]$$

$$+ [2nJ_3(1,n,p,1) - (1+n^2)\ J_1(1,1,n,p)][(1+q^2)J_3(1,m,q,1) - 2qJ_1(1,1,q,m)]$$

$$+ [2nJ_1(1,n,p,1) + (1+n^2)\ J_1(1,1,p,n)][2qJ_1(1,1,m,q) + (1+q^2)\ J_1(1,m,q,1)]$$

$$+ [2nI_2(1,n,p,1) + (1+n^2)I_2(1,1,n,p)][(1+q^2)\ I_2(1,m,q,1) - 2qI_0(1,1,m,q)]\}$$

$$G^{55}_{44nmpq} = \int_0^1 \int_0^1 f_{4,5}(x,y;n,m) f_{4,5}(x,y;p,q)\,dxdy$$

$$= \pi^4\{[I_2(n,p,1,1) + nI_2(1,p,n,1) + pI_2(1,n,p,1) + npI_2(1,1,n,p)]$$
$$[I_4(m,q,1,1) - mI_2(1,m,q,1) - qI_2(1,q,m,1) + mqI_0(1,1,m,q)]$$

$$- [J_3(p,n,1,1) - nJ_1(1,n,p,1) + pJ_3(1,n,p,1) - npJ_1(1,1,n,p)]$$
$$[J_3(m,q,1,1) + mJ_3(1,m,q,1) - qJ_1(1,m,q,1) - mqJ_1(1,1,q,m)]$$

$$- [J_3(n,p,1,1) + nJ_3(1,n,p,1) - pJ_1(1,n,p,1) - npJ_1(1,1,p,n)]$$
$$[J_3(q,m,1,1) - mJ_1(1,m,q,1) + qJ_3(1,m,q,1) - mqJ_1(1,1,m,q)]$$

$$+ [I_4(n,p,1,1) - nI_2(1,n,p,1) - pI_2(1,p,n,1) + npI_0(1,1,n,p)]$$
$$[I_2(m,q,1,1) + mI_2(1,q,m,1) + qI_2(1,m,q,1) + mqI_2(1,1,m,q)]\}$$

$$G^{00}_{34nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_4(x,y;p,q)\,dxdy = J_0(1,n,p)\ I_1(1,n,p)\ J_0(1,m,q)$$

$$G^{00}_{44nmpq} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_4(x,y;p,q)\,dxdy$$

$$= I_0(1,1,n,p)\ I_2(1,1,m,q) - J_1(1,1,n,p)\ J_1(1,1,q,m)$$
$$- J_1(1,1,p,n)\ J_1(1,1,m,q) + I_2(1,1,n,p)\ I_0(1,1,m,q)$$

$$G^{10}_{13nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_3(x,y;p,q)\,dxdy$$

$$= \pi\{[J_2(p,n,1) - nJ_0(1,n,p)]\ J_1(q,m)$$

$$+ [I_1(n,p,1) + nI_1(1,p,n)]\ I_0(m,q)\}$$

$$G^{10}_{14nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_4(x,y;p,q)\,dxdy$$

$$= \pi\{[I_2(1,p,n,1) - nI_0(1,1,n,p)]\ J_2(1,m,q)$$

$$- [J_3(1,n,p,1) - nJ_1(1,1,n,p)]\ I_1(1,q,m)$$

$$+ [J_1(1,n,p,1) + nJ_1(1,1,p,n)]\ I_1(1,m,q)$$

$$- [I_2(1,n,p,1) + nI_2(1,1,n,p)]\ J_0(1,m,q)\}$$

$$G^{11}_{11nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{1,1}(x,y;p,q)\,dxdy$$

$$= \pi^2\{[I_4(n,p,1,1) + I_2(n,p,1,1) - (n-p)(I_2(1,n,p,1)$$

$$- I_2(1,p,n,1) + np(I_0(1,1,n,p) + I_2(1,1,n,p)]\ I_0(m,q)$$

$$+ [J_3(n,p,1,1) + nJ_3(1,n,p,1) - pJ_1(1,n,p,1)$$

$$- npJ_1(1,1,p,n)]\ J_1(m,q)$$

$$+ [J_3(p,n,1,1) - nJ_1(1,n,p,1) + pJ_3(1,n,p,1)$$

$$- npJ_1(1,1,n,p)]\ J_1(q,m)\}$$

$$G^{12}_{12nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{2,2}(x,y;p,q)\,dxdy$$

$$= q\pi^2\{[J_2(p,n,1) - J_2(n,p,1) - J_3(p,n,1,1) + J_3(n,p,1,1) - n(J_0(1,n,p)$$

$$+ J_2(1,n,p) - J_1(1,n,p,1) - J_3(1,n,p,1))]\ I_0(m,q)$$

$$+ [I_1(n,p,1) - I_2(n,p,1,1) + n(I_1(1,p,n) - I_2(1,p,n,1))]\ J_1(m,q)$$

$$- [I_3(n,p,1) - I_4(n,p,1,1) - n(I_1(1,n,p) - I_2(1,n,p,1))]\ J_1(q,m)\}$$

$$G^{22}_{11nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{1,2}(x,y;p,q)\,dxdy$$

$$= mq\pi^2\{[I_0(1,1,n,p) + I_2(1,1,n,p)]\ I_0(m,q)$$

$$- J_1(1,1,n,p)\ J_1(q,m) - J_1(1,1,p,n)\ J_1(m,q)\}$$

$$G^{21}_{12nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{2,1}(x,y;p,q)\,dxdy$$

$$= m\pi^2\{[J_1(1,1,n,p) - J_1(1,1,p,n) + p(J_1(1,n,p,1) + J_3(1,n,p,1)$$

$$J_0(1,n,p) - J_2(1,n,p))]\ I_0(m,q)$$

$$+ [I_0\ (1,1,n,p) + p(I_1(1,n,p) - I_2(1,n,p,1))]\ J_1(q,m)$$

$$- [I_2(1,1,n,p) - p(I_1(1,p,n) - I_2(1,p,n,1))]\ J_1(m,q)\}$$

$$G^{11}_{22nmpq} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{2,1}(x,y;p,q)\,dxdy$$

$$= \pi^2\{[I_2(1,1,n,p) + I_0(1,1,n,p) + (n-p)(I_2(1,n,p,1) - I_2(1,p,n,1)$$

$$- I_1(1,n,p) + I_1(1,p,n)) + np(I_2(n,p,1,1) + I_4(n,p,1,1)$$

$$- 2I_1(n,p,1) - 1I_3(n,p,1) + 2I_0(n,p))]\ I_0(m,q)$$

$$+ [J_1(1,1,p,n) + n(J_1(1,n,p,1) - J_0(1,n,p)) - p(J_3(1,n,p,1)$$

$$- J_2(1,n,p)) - np(J_3(n,p,1,1) - 1J_2(n,p,1) + J_1(n,p))]\ J_1(q,m)$$

$$+ [J_1(1,1,n,p) - n(J_3(1,n,p,1) - J_2(1,n,p)) + p(J_1(1,n,p,1)$$

$$- J_0(1,n,p)) - np(J_3(p,n,1,1) - 2J_2(p,n,1) + J_1(p,n))]\ J_1(m,q)\}$$

$$G^{20}_{23nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_3(x,y;p,q)\,dxdy$$

$$= m\pi\{[I_0(n,p) - I_1(n,p,1)]\ J_1(q,m)$$

$$[J_1(p,n) - J_2(p,n,1)]\ I_0(m,q)\}$$

$$G^{20}_{24nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_4(x,y;p,q)\,dxdy$$

$$= m\pi\{[J_0(1,n,p) - J_1(1,n,p,1)]\ J_2(1,m,q)$$

$$- [I_1(1,n,p) - I_2(1,n,p,1)]\ I_1(1,q,m)$$

$$- [I_1(1\ \ n) - I_2(1,p,n,1)]\ I_1(1,m,q)$$

$$+ [J_2(1,n,p) - J_3(1,n,p,1)]\ J_0(1,m,q)\}$$

$$G^{22}_{22nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{2,2}(x,y;p,q)\,dxdy$$

$$= mq\pi^2\{[2I_0(n,p) - 2I_1(n,p,1) - 2I_3(n,p,1) + I_2(n,p,1,1)$$

$$+ I_4(n,p,1,1)]\ I_0(m,q)$$

$$- [J_1(n,p) - 2J_2(n,p,1) + J_3(n,p,1,1)]\ J_1(q,m)$$

$$- [J_1(p,n) - 2J_2(p,n,1) + J_3(p,n,1,1)]\ J_1(m,q)\}$$

$$H^{110}_{335nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_5(x,y)\,dxdy$$

$$= np\pi^2\ J_2(1,n,p)\ J_0(1,m,q)$$

$$H^{220}_{335nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_5(x,y)\,dxdy = H^{110}_{335mnqp}$$

$$H^{110}_{345nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_5(x,y)\,dxdy$$

$$= n\pi^2\ \{[I_2(1,p,n,1) + pI_2(1,1,n,p)]\ J_1(1,1,m,q)$$

$$- [J_3(1,n,p,1) - pJ_1(1,1,p,n)]\ I_0(1,1,m,q)\}$$

$$H^{220}_{345nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_5(x,y)\,dxdy = -\ H^{110}_{345mnqp}$$

$$H^{101}_{345nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_4(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^2 \,[I_2(1,p,n,1)\ J_1(1,1,m,q) - J_3(1,n,p,1) I_0(1,1,m,q)]$$

$$H^{202}_{345nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_4(x,y;p,q) f_{5,2}(x,y)\,dxdy = -\ H^{101}_{345mnqp}$$

$$H^{110}_{445nmpq} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_5(x,y)\,dxdy$$

$$= \pi^2 \,[U_6(1,n,p)\ J_2(1,1,1,m,q) - U_4(1,p,n) I_1(1,1,1,q,m)$$

$$- U_4(1,n,p)\ I_1(1,1,1,m,q) + U_2(1,n,p)\ J_0(1,1,1,m,q)]$$

$$H^{220}_{445nmpq} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_5(x,y)\,dxdy = H^{110}_{445mnqp}$$

$$H^{011}_{335nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^2\ J_2(n,p,1)\ J_0(1,m,q)$$

$$H^{022}_{335nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{3,2}(x,y;pq) f_{5,2}(x,y)\,dxdy = H^{011}_{335mnqp}$$

$$H^{011}_{345nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= \pi^2 \,\{[I_2(n,p,1,1) + pI_2(1,n,p,1)]\ J_1(1,1,m,q)$$

$$- [J_3(n,p,1,1) - pJ_1(1,n,p,1)]\ I_0(1,1,m,q)\}$$

$$H^{022}_{345nmpq} = \int_0^1 \int_0^1 f_3(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy = -\ H^{011}_{345mnqp}$$

$$H^{011}_{445nmpq} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= \pi^2 \{[J_2(1,n,p,1,1) + pJ_2(1,1,n,p,1)]\ J_2(1,1,1,m,q)$$

$$- [I_3(1,n,p,1,1) - pI_1(1,1,n,p,1)]\ I_1(1,1,1,q,m)$$

$$- [I_3(1,p,n,1,1) + pI_3(1,1,n,p,1)]\ I_1(1,1,1,m,q)$$

$$+ [J_4(1,n,p,1,1) - pJ_2(1,1,p,n,1)]\ J_0(1,1,1,m,q)\}$$

$$H^{022}_{445nmpq} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy = H^{011}_{445mnqp}$$

$$H^{111}_{135nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= p\pi^3 \{[I_4(n,p,1,1) - nI_2(1,n,p,1)]\ I_1(1,q,m)$$

$$+ [J_3(n,p,1,1) + nJ_3(1,n,p,1)]\ J_0(1,m,q)\}$$

$$H^{122}_{135nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= q\pi^3 \{[I_2(1,p,n,1) - nI_0(1,1,n,p)]\ I_3(m,q,1)$$

$$+ [J_1(1,n,p,1) + nJ_1(1,1,p,n)]\ J_2(m,q,1)\}$$

$$H^{111}_{145nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= \pi^3 \{[U_3(1,n,p)\ I_2(1,1,m,q) - U_1(1,n,p)\ J_1(1,1,q,m)$$

$$+ U_5(1,n,p)\ J_1(1,1,m,q) - U_3(1,p,n)\ I_0(1,1,m,q)]$$

$$H^{122}_{145nmpq} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= \pi^3 \{[J_2(1,1,p,n,1) - nJ_0(1,1,1,n,p)][I_4(m,q,1,1) - qI_2(1,q,m,1)]$$

$$- [I_3(1,1,n,p,1) - nI_1(1,1,1,n,p)][J_3(q,m,1,1) + qJ_3(1,m,q,1)]$$

$$+ [I_1(1,1,n,p,1) + nI_1(1,1,1,p,n)][J_3(m,q,1,1) - qJ_1(1,m,q,1)]$$

$$- [J_2(1,1,n,p,1) + nJ_2(1,1,1,n,p)][I_2(m,q,1,1) + qI_2(1,m,q,1)]\}$$

$$H^{212}_{135nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= mp\pi^3 [J_1(1,1,n,p)\ J_2(q,m,1) - I_2(1,1,n,p)\ I_1(m,q,1)]$$

$$H^{221}_{135nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= mq\pi^3 [J_1(1,n,p,1)\ J_2(1,m,q) - I_2(1,p,n,1)\ I_1(1,m,q)]$$

$$H^{212}_{145nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= m\pi^3 \{[I_1(1,1,n,p,1) + pI_1(1,1,1,n,p)]\ J_3(1,m,q,1)$$

$$- [J_2(1,1,n,p,1) - pJ_0(1,1,1,n,p)]\ I_2(1,q,m,1)$$

$$- [J_2(1,1,p,n,1) + pJ_2(1,1,1,n,p)]\ I_2(1,m,q,1)$$

$$+ [I_3(1,1,n,p,1) - pI_1(1,1,1,p,n)]\ J_1(1,m,q,1)\}$$

$$H^{221}_{145nmpq} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= m\pi^3 \{I_1(1,1,n,p,1)\ [J_3(1,m,q,1) - qJ_1(1,1,q,m)]$$

$$- J_2(1,1,n,p,1)\ [I_2(1,q,m,1) + qI_2(1,1,m,q)]$$

$$- J_2(1,1,p,n,1)\ [J_2(1,m,q,1) - qI_0(1,1,m,q)]$$

$$+ J_3(1,1,n,p,1)\ [J_1(1,m,q,1) + qJ_1(1,1,m,q)]\}$$

$$
\begin{aligned}
H^{112}_{235nmpq} &= \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,2}(x,y)\,dxdy \\
&= p\pi^3 \{[n(I_2(1,n,p,1) - I_1(1,n,p)) + I_2(1,1,n,p)]\ J_2(q,m,1) \\
&\qquad + [n(J_2(1,n,p) - J_3(1,n,p,1)) + J_1(1,1,n,p)]\ I_1(m,q,1)\}
\end{aligned}
$$

$$
\begin{aligned}
H^{121}_{235nmpq} &= \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,1}(x,y)\,dxdy \\
&= q\pi^3 \{[n(I_2(n,p,1,1) - I_1(n,p,1)) + I_2(1,p,n,1)]\ J_2(1,m,q) \\
&\qquad + [n(J_2(p,n,1) - J_3(p,n,1,1)) + J_1(1,n,p,1)]\ I_1(1,m,q)\}
\end{aligned}
$$

$$
\begin{aligned}
H^{112}_{245nmpq} &= \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,2}(x,y)\,dxdy \\
&= \pi^3 \{[J_2(1,1,p,n,1) + pJ_2(1,1,1,n,p) + n(J_2(1,n,p,1,1) \\
&\qquad - J_1(1,n,p,1)) + np(J_2(1,1,n,p,1) - J_1(1,1,n,p))]\ J_3(1,m,q,1) \\
&\quad - [I_3(1,1,n,p,1) - pI_1(1,1,1,p,n) + n(I_3(1,n,p,1,1) - I_2(1,n,p,1)) \\
&\qquad - np(I_1(1,1,n,p,1) - I_0(1,1,n,p))]\ I_2(1,q,m,1) \\
&\quad + [I_1(1,1,n,p,1) + pI_1(1,1,1,n,p) - n(I_3(1,p,n,1,1) - I_2(1,p,n,1)) \\
&\qquad - np(I_3(1,1,n,p,1) - I_2(1,1,n,p))]\ I_2(1,m,q,1) \\
&\quad - [J_2(1,1,n,p,1) - pJ_0(1,1,1,n,p) - n(J_4(1,n,p,1,1) - J_3(1,n,p,1)) \\
&\qquad + np(J_2(1,1,p,n,1) - J_1(1,1,p,n))]\ J_1(1,m,q,1)\}
\end{aligned}
$$

$$
\begin{aligned}
H^{121}_{245nmpq} &= \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,1}(x,y)\,dxdy \\
&= \pi^3 \{[J_2(1,1,p,n,1) + n(J_2(1,n,p,1,1) - J_1(1,n,p,1))] \\
&\qquad\qquad [J_3(1,m,q,1) - qJ_1(1,1,q,m)] \\
&\qquad - [I_3(1,1,n,p,1) + n(I_3(1,n,p,1,1) - I_2(1,n,p,1))] \\
&\qquad\qquad [I_2(1,q,m,1) + qI_2(1,1,m,q)] \\
&\qquad + [I_1(1,1,n,p,1) - n(I_3(1,p,n,1,1) - I_2(1,p,n,1))]
\end{aligned}
$$

$$[I_2(1,m,q,1) - qI_0(1,1,m,q)]$$

$$- [J_2(1,1,n,p,1) - n(J_4(1,n,p,1,1) - J_3(1,n,p,1))]$$

$$[J_1(1,m,q,1) + qJ_1(1,1,m,q)]\}$$

$$H^{211}_{235nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= mp\pi^3 \{[J_2(n,p,1) - J_3(n,p,1,1)]\ I_1(1,q,m)$$

$$- [I_3(n,p,1) - J_4(n,p,1,1)]\ J_0(1,m,q)\}$$

$$H^{222}_{235nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= mq\pi^3 \{[J_0(1,n,p) - J_1(1,n,p,1)]\ I_3(m,q,1)$$

$$- [I_1(1,p,n) - I_2(1,p,n,1)]\ J_2(m,q,1)\}$$

$$H^{211}_{245nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y)\,dxdy$$

$$= m\pi^3 \{[I_2(n,p,1,1) - I_3(n,p,1,1,1) + p(I_2(1,n,p,1)$$

$$- I_3(1,n,p,1,1))]\ I_2(1,1,m,q)$$

$$- [J_3(n,p,1,1) - J_4(n,p,1,1,1) - p(J_1(1,n,p,1)$$

$$- J_2(1,n,p,1,1))]\ J_1(1,1,q,m)$$

$$- [J_3(p,n,1,1) - J_4(p,n,1,1,1) + p(J_3(1,n,p,1)$$

$$- J_4(1,n,p,1,1))]\ J_1(1,1,m,q)$$

$$+ [I_4(n,p,1,1) - I_5(n,p,1,1,1) - p(I_2(1,p,n,1)$$

$$- I_3(1,p,n,1,1))]\ I_0(1,1,m,q)\}$$

$$H^{222}_{245nmpq} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y)\,dxdy$$

$$= m\pi^3 \{[I_0(1,1,n,p) - I_1(1,1,n,p,1)][I_4(m,q,1,1) - qI_2(1,q,m,1)]$$

$$- [J_1(1,1,n,p) - J_2(1,1,n,p,1)][J_3(q,m,1,1) + qJ_3(1,m,q,1)]$$

$$- [J_1(1,1,p,n) - J_2(1,1,p,n,1)][J_3(m,q,1,1) - qJ_1(1,m,q,1)]$$

$$+ [I_2(1,1,n,p) - I_3(1,1,n,p,1)][I_2(m,q,1,1) + qI_2(1,m,q,1)]\}$$

$$I^{1111}_{3355nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= np\pi^4\, I_4(n,p,1,1)\, I_0(1,1,m,q)$$

$$I^{1122}_{3355nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= np\pi^4\, I_2(1,1,n,p)\, I_2(m,q,1,1)$$

$$I^{1212}_{3355nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= nq\pi^4\, I_2(1,p,n,1)\, I_2(1,m,q.1)$$

$$I^{1111}_{3455nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^4 \{[J_4(p,n,1,1,1) + pJ_4(1,n,p,1,1)]\, I_1(1,1,1,m,q)$$

$$- [I_5(n,p,1,1,1) - pI_3(1,p,n,1,1)]\, J_0(1,1,1,m,q)\}$$

$$I^{1122}_{3455nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= n\pi^4 \{[J_2(1,1,p,n,1) + pJ_2(1,1,1,n,p)]\, I_3(1,m,q,1,1)$$

$$- [I_3(1,1,n,p,1) - pI_1(1,1,1,p,n)]\, J_2(1,m,q,1,1)\}$$

$$I^{1212}_{3455nmpq} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= n\pi^4 \{J_2(1,1,p,n,1)\ [I_3(1,m,q,1,1) - qI_1(1,1,m,q,1)]$$

$$- I_3(1,1,n,p,1)\ [J_2(1,m,q,1,1) + qJ_2(1,1,m,q,1)]\}$$

$$I^{2211}_{3355nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy = I^{1122}_{3355mnqp}$$

$$I^{2222}_{3355nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;pq) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy = I^{1111}_{3355mnqp}$$

$$I^{2112}_{3455nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= - I^{1212}_{3455mnqp}$$

$$I^{2222}_{3455nmpq} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= - I^{1111}_{3455mnqp}$$

$$I^{1111}_{4455nmpq} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= \pi^4\ [B_1(1,1,n,p)\ I_2(1,1,1,1,m,q) - B_2(1,1,p,n)\ J_1(1,1,1,1,q,m)$$

$$- B_2(1,1,n,p)\ J_1(1,1,1,1,m,q) + B_3(1,1,n,p)\ I_0(1,1,1,1,m,q)]$$

$$I^{1122}_{4455nmpq} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= \pi^4\ [B_6(1,1,n,p)\ I_4(1,1,m,q,1,1) - B_5(1,1,p,n)\ J_3(1,1,q,m,1,1)$$

$$- B_5(1,1,n,p)\ J_3(1,1,m,q,1,1) + B_4(1,1,n,p)\ I_2(1,1,m,q,1,1)]$$

$$I^{1212}_{4455nmpq} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,1}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= \pi^4 \,[C_1(1,1,p,n)\; C_4(1,1,m,q) - C_3(1,p,1,n)\; C_3(1,m,1,q)$$

$$- C_2(1,1,p,n)\; C_2(1,1,m,q) + C_4(1,1,p,n)\; C_1(1,1,m,q)]$$

$$I^{2211}_{4455nmpq} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,1}(x,y) f_{5,1}(x,y)\,dxdy$$

$$= I^{1122}_{3455mnqp}$$

$$I^{2222}_{4455nmpq} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{5,2}(x,y) f_{5,2}(x,y)\,dxdy$$

$$= I^{1111}_{4455mnqp}$$

$$H^{110}_{334nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_4(x,y;r,s)\,dxdy$$

$$= np\pi^2 \,[I_2(1,r,n,p)\; J_1(1,m,q,s) - J_3(1,n,p,r)\; I_0(1,r,q,s)]$$

$$H^{110}_{344nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_4(x,y;r,s)\,dxdy$$

$$= n\pi^2 \,\{[J_2(1,p,r,n,1) + pJ_2(1,1,r.n,p)]\; J_2(1,1,m,q,s)$$

$$- [I_3(1,p,n,r,1) + pI_3(1,1,n,p,r)]\; I_1(1,1,m,s,q)$$

$$- [I_3(1,r,n,p,1) - pI_1(1,1,p,r,n)]\; I_1(1,1,m,q,s)$$

$$+ [J_4(1,n,p,r,1) - pJ_2(1,1,p,n,r)]\; J_0(1,1,m,q,s)\}$$

$$H^{220}_{334nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_4(x,y;r,s)\,dxdy = - H^{110}_{334mnqpsr}$$

$$H^{202}_{344nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m)f_4(x,y;p,q)f_{4,2}(x,y;r,s)dxdy = H^{110}_{344mnsrqp}$$

$$H^{011}_{333nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{3,1}(x,y;p,q)f_{3,1}(x,y;r,s)dxdy$$

$$= pr\pi^2\ J_2(n,p,r)\ J_0(m,q,s)$$

$$H^{011}_{334nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{3,1}(x,y;p,q)f_{4,1}(x,y;r,s)dxdy$$

$$= p\pi^2\ \{[I_2(n,r,p,1) + rI_2(1,n,p,r)]\ J_1(1,m,q,s)$$

$$- [J_3(n,p,r,1) - rJ_1(1,n,r,p)]\ I_0(1,m,q,s)\}$$

$$H^{022}_{333nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{3,2}(x,y;p,q)f_{3,2}(x,y;r,s)dxdy = H^{011}_{333mnqpsr}$$

$$H^{022}_{334nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{3,2}(x,y;p,q)f_{4,2}(x,y;r,s)dxdy = -\ H^{011}_{334mnqpsr}$$

$$H^{011}_{344nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{4,1}(x,y;p,q)f_{4,1}(x,y;r,s)dxdy$$

$$= \pi^2\ [U_6(n,p,r)\ J_2(1,1,m,q,s) - U_4(n,r,p)\ I_1(1,1,m,s,q)$$

$$- U_4(n,p,r)\ I_1(1,1,m,q,s) + U_2(n,p,r)\ J_0(1,1,m,q,s)]$$

$$H^{022}_{344nmpqrs} = \int_0^1 \int_0^1 f_3(x,y;n,m)f_{4,2}(x,y;p,q)f_{4,2}(x,y;r,s)dxdy = H^{011}_{344mnqpsr}$$

$$H^{011}_{444nmpqrs} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s)\,dxdy$$

$$= \pi^2 \,[B_6(1,n,p,r)\, J_3(1,1,1,m,q,s) - B_5(1,n,r,p)\, I_2(1,1,1,s,m,q)$$

$$- B_5(1,n,p,r)\, I_2(1,1,1,q,m,s) + B_4(1,n,p,r)\, J_1(1,1,1,q,s,m)$$

$$- V_6(n,p,r)\, I_2(1,1,1,m,q,s) + V_5(n,r,p)\, J_1(1,1,1,m,s,q)$$

$$+ V_5(n,p,r)\, J_1(1,1,1,m,q,s) - V_4(n,p,r)\, I_0(1,1,1,m,q,s)]$$

$$H^{022}_{444nmpqrs} = \int_0^1 \int_0^1 f_4(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy = - H^{011}_{444mnqpsr}$$

$$H^{111}_{133nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,1}(x,y;r,s)\,dxdy$$

$$= pr\pi^3 \,\{[I_4(n,p,r,1) - nI_2(1,n,p,r)]\, I_1(q,s,m)$$

$$+ [J_3(n,p,r,1) + nJ_3(1,n,p,r)]\, J_0(m,q,s)\}$$

$$H^{111}_{134nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,1}(x,y;r,s)\,dxdy$$

$$= p\pi^3 \,[U_3(p,n,r)\, I_2(1,q,m,s) - U_1(p,n,r)\, J_1(1,q,s,m)$$

$$+ U_5(p,n,r)\, J_1(1,m,q,s) - U_3(p,r,n)\, I_0(1,m,q,s)]$$

$$H^{122}_{133nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s)\,dxdy$$

$$= qs\pi^3 \,\{[I_2(p,r,n,1) - nI_0(1,n,p,r)]\, I_3(m,q,s)$$

$$+ [J_1(n,p,r,1) + nJ_1(1,p,r,n)]\, J_2(m,q,s)\}$$

$$H^{122}_{134nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy$$

$$= q\pi^3 \{[J_2(1,p,r,n,1) - nJ_0(1,1,n,p,r)][I_4(m,q,s,1) - sI_2(1,s,m,q)]$$

$$- [I_3(1,p,n,r,1) - nI_1(1,1,n,p,r)][J_3(s,m,q,1) + sJ_3(1,m,q,s)]$$

$$+ [I_1(1,n,p,r,1) + nI_1(1,1,p,r,n)][J_3(m,q,s,1) - sJ_1(1,m,s,q)]$$

$$- [J_2(1,n,p,r,1) + nJ_2(1,1,p,n,r)][I_2(m,s,q,1) + sI_2(1,m,q,s)]\}$$

$$H^{111}_{144nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s)\,dxdy$$

$$= \pi^3 [W_2(n,p,r)\ I_3(1,1,m,q,s) - W_3(n,r,p)\ J_2(1,1,s,m,q)$$

$$- W_3(n,p,r)\ J_2(1,1,q,m,s) + W_1(n,p,r)\ I_1(1,1,q,s,m)$$

$$+ W_4(n,p,r)\ J_2(1,1,m,q,s) - W_2(r,n,p)\ I_1(1,1,m,s,q)$$

$$- W_2(p,n,r)\ I_1(1,1,m,q,s) + W_3(p,r,n)\ J_0(1,1,m,q,s)]$$

$$H^{122}_{144nmpqrs} = \int_0^1 \int_0^1 f_{1,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy$$

$$= \pi^3 [T_1(n,p,r)\ U_1(m,q,s) - C_2(1,r,p,n)\ U_3(m,q,s)$$

$$- C_2(1,p,r,n)\ U_3(m,s,q) + C_4(1,r,p,n)\ U_5(m,q,s)$$

$$+ T_2(n,p,r)\ U_2(m,q,s) - C_1(1,r,p,n)\ U_4(m,q,s)$$

$$- C_1(1,p,r,n)\ U_4(m,s,q) + C_3(1,p,r,n)\ U_6(m,q,s)]$$

$$H^{212}_{133nmpqrs} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,2}(x,y;r,s)\,dxdy$$

$$= mps\pi^3 [J_1(1,n,r,p)\ J_2(q,m,s) - I_2(1,r,n,p)\ I_1(m,q,s)]$$

$$H^{212}_{134nmpqrs} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy$$

$$\begin{aligned} = mp\pi^3 \{ & I_1(1,1,n,r,p)\ [J_3(q,m,s,1) - sJ_1(1,q,s,m)] \\ & - J_2(1,1,n,p,r)\ [J_2(q,s,m,1) + sI_2(1,q,m,s)] \\ & - J_2(1,1,r,n,p)\ [I_2(m,q,s,1) - sI_0(1,m,q,s)] \\ & + I_3(1,1,n,p,r)\ [J_1(m,q,s,1) + sJ_1(1,m,q,s)]\} \end{aligned}$$

$$H^{221}_{134nmpqrs} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,1}(x,y;r,s)\,dxdy$$

$$\begin{aligned} = mq\pi^3 \{ & [I_1(1,n,p,r,1) + rI_1(1,1,n,p,r)]\ J_3(1,m,q,s) \\ & - [J_2(1,n,p,r,1) - rJ_0(1,1,n,p,r)]\ I_2(1,s,m,q) \\ & - [J_2(1,p,r,n,1) + rJ_2(1,1,p,n,r)]\ I_2(1,m,q,s) \\ & + [I_3(1,p,n,r,1) - rI_1(1,1,p,r,n)]\ J_1(1,m,s,q)\} \end{aligned}$$

$$H^{212}_{144nmpqrs} = \int_0^1 \int_0^1 f_{1,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy$$

$$\begin{aligned} = m\pi^3 \{ & T_2(p,n,r)\ [J_4(1,m,q,s,1) - sJ_2(1,1,s,m,q)] \\ & - C_1(1,r,n,p)\ [I_3(1,s,m,q,1) + sI_3(1,1,m,q,s)] \\ & - T_1(p,n,r)\ [I_3(1,q,m,s,1) - sI_1(1,1,q,s,m)] \\ & + C_2(1,r,n,p)\ [J_2(1,q,s,m,1) + sJ_2(1,1,q,s,m)] \\ & - C_1(1,n,r,p)\ [I_3(1,m,q,s,1) - sI_1(1,1,m,s,q)] \\ & + C_3(1,n,r,p)\ [J_2(1,m,s,q,1) + sJ_2(1,1,m,q,s)] \\ & + C_2(1,n,r,p)\ [J_2(1,m,q,s,1) - sJ_0(1,1,m,q,s)] \\ & - C_4(1,r,n,p)\ [I_1(1,m,q,s,1) + sI_1(1,1,m,q,s)]\} \end{aligned}$$

$$H^{112}_{233nmpqrs} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m)f_{3,1}(x,y;p,q)f_{3,2}(x,y;r,s)dxdy$$

$$= ps\pi^3 \{[I_2(1,r,n,p) + n(I_2(n,r,p,1) - I_1(n,r,p))] J_2(q,m,s)$$
$$+ [J_1(1,n,r,p) - n(J_3(r,n,p,1) - J_2(r,n,p))] I_1(m,q,s)\}$$

$$H^{112}_{234nmpqrs} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m)f_{3,1}(x,y;p,q)f_{4,2}(x,y;r,s)dxdy$$

$$= p\pi^3 \{[J_2(1,1,r,n,p) + n(J_2(1,n,r,p,1) - J_1(1,n,r,p))]$$
$$[J_3(q,m,s,1) - sJ_1(1,q,s,m)]$$
$$- [I_3(1,1,n,p,r) + n(I_3(1,n,p,r,1) - I_2(1,n.p,r))]$$
$$[I_2(q,s,m,1) + sI_2(1,q,m,s)]$$
$$+ [I_1(1,1,n,r,p) - n(I_3(1,r,n,p,1) - I_2(1,r,n,p))]$$
$$[I_2(m,q,s,1) - sI_0(1,m,q,s)]$$
$$- [J_2(1,1,n,p,r) - n(J_4(1,n,p,r,1) - J_3(1,n,p,r))]$$
$$[J_1(m,q,s,1) + sJ_1(1,m,q,s)]\}$$

$$H^{121}_{234nmpqrs} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m)f_{3,2}(x,y;p,q)f_{4,1}(x,y;r,s)dxdy$$

$$= q\pi^3 \{[J_2(1,p,r,n,1) + n(J_2(n,p,r,1,1) - J_1(n,p,r,1)) + rJ_2(1,1,p,n,r)$$
$$+ nr(J_2(1,n,p,r,1) - J_1(1,n,p,r))] J_3(1,m,q,s)$$
$$- [I_3(1,p,n,r,1) + n(I_3(n,p,r,1,1) - I_2(n,p,r,1)) - rI_1(1,1,p,r,n)$$
$$- nr(I_1(1,n,r,p,1) - I_0(1,n,r,p))] I_2(1,s,m,q)$$
$$+ [I_1(1,n,p,r,1) - n(I_3(p,r,n,1,1) - I_2(p,r,n,1)) + rI_1(1,1,n,p,r)$$
$$- nr(I_3(1,p,n,r,1) - I_2(1,p,n,r))] I_2(1,m,q,s)$$
$$- [J_2(1,n,p,r,1) - n(J_4(p,n,r,1,1) - J_3(p,n,r,1)) - rJ_0(1,1,n,p,r)$$
$$+ nr(J_2(1,p,r,n,1) - J_1(1,p,r,n))] J_1(1,m,s,q)\}$$

$$H^{112}_{244nmpqrs} = \int_0^1 \int_0^1 f_{2,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s)\,dxdy$$

$$= \pi^3 \{Q_1(n,p,r)\ [J_4(1,m,q,s,1) - sJ_2(1,1,s,m,q)]$$

$$- Q_2(n,p,r)\ [I_3(1,s,m,q,1) + sI_3(1,1,m,q,s)]$$

$$- Q_3(n,p,r)\ [I_3(1,q,m,s,1) - sI_1(1,1,q,s,m)]$$

$$+ Q_4(n,p,r)\ [J_2(1,q,s,m,1) + sJ_2(1,1,q,m,s)]$$

$$+ Q_5(n,p,r)\ [I_3(1,m,q,s,1) - sI_1(1,1,m,s,q)]$$

$$- Q_6(n,p,r)\ [J_2(1,m,s,q,1) + sJ_2(1,1,m,q,s)]$$

$$- Q_7(n,p,r)\ [J_2(1,m,q,s,1) - sJ_0(1,1,m,q,s)]$$

$$+ Q_8(n,p,r)\ [I_1(1,m,q,s,1) + sI_1(1,1,m,q,s)]\}$$

$$H^{211}_{233nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,1}(x,y;r,s)\,dxdy$$

$$= mpr\pi^3 \{[J_2(n,p,r) - J_3(n,p,r,1)]\ I_1(q,s,m)$$

$$- [I_3(n,p,r) - I_4(n,p,r,1)]\ J_0(m,q,s)\}$$

$$H^{211}_{234nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,1}(x,y;r,s)\,dxdy$$

$$= mp\pi^3 \{[I_2(n,r,p,1) - I_3(n,r,p,1,1) + r(I_2(1,n,r,p)$$

$$- I_3(1,n,r,p,1))]\ I_2(1,q,m,s)$$

$$- [J_3(n,p,r,1) - J_4(n,p,r,1,1) - r(J_1(1,n,r,p)$$

$$- J_2(1,n,r,p,1))]\ J_1(1,q,s,m)$$

$$- [J_3(r,n,p,1) - J_4(r,n,p,1,1) + r(J_3(1,n,p,r)$$

$$- J_4(1,n,p,r,1))]\ J_1(1,m,q,s)$$

$$+ [I_4(n,p,r,1) - I_5(n,p,r,1,1) - r(I_2(1,r,n,p)$$

$$- I_3(1,r,n,p,1)]\ I_0(1,m,q,s)\}$$

$$H^{222}_{233nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)f_{3,2}(x,y;p,q)f_{3,2}(x,y;r,s)dxdy$$

$$= mqs\pi^3 \{[J_0(n,p,r) - J_1(n,p,r,1)]\ I_3(m,q,s)$$

$$- [I_1(p,r,n) - I_2(p,r,n,1)]\ J_2(m,q,s)\}$$

$$H^{222}_{234nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)f_{3,2}(x,y;p,q)f_{4,2}(x,y;r,s)dxdy$$

$$= mq\pi^3 \{[I_0(1,n,r,p) - I_1(1,n,r,p,1)][I_4(m,q,s,1) - sI_2(1,s,m,q)]$$

$$- [J_1(1,n,p,r) - J_2(1,n,p,r,1)][J_3(s,m,q,1) + SJ_3(1,m,q,s)]$$

$$- [J_1(1,p,r,n) - J_2(1,p,r,n,1)][J_3(m,q,s,1) - sJ_1(1,m,s,q)]$$

$$+ [I_2(1,p,n,r) - I_3(1,p,n,r,1)][I_2(m,s,q,1) + sI_2(1,m,q,s)]\}$$

$$H^{211}_{244nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)f_{4,1}(x,y;p,q)f_{4,1}(x,y;r,s)dxdy$$

$$= m\pi^3\ [P_1(n,p,r)\ I_3(1,1,m,q,s) - P_2(n,r,p)\ J_2(1,1,s,m,q)$$

$$- P_2(n,p,r)\ J_2(1,1,q,m,s) + P_3(n,p,r)\ I_1(1,1,q,s,m)$$

$$- P_4(n,p,r)\ J_2(1,1,m,q,s) + P_5(n,r,p)\ I_1(1,1,m,s,q)$$

$$+ P_5(n,p,r)\ J_1(1,1,m,q,s) - P_6(n,p,r)\ J_0(1,1,m,q,s)]$$

$$H^{222}_{244nmpqrs} = \int_0^1 \int_0^1 f_{2,2}(x,y;n,m)f_{4,2}(x,y;p,q)f_{4,2}(x,y;r,s)dxdy$$

$$= m\pi^3 \{[J_0(1,1,n,p,r) - J_1(1,1,n,p,r,1)]\ U_1(m,q,s)$$

$$- [I_1(1,1,n,p,r) - I_2(1,1,n,p,r,1)]\ U_3(m,q,s)$$

$$- [I_1(1,1,n,r,p) - I_2(1,1,n,r,p,1)]\ U_3(m,s,q)$$

$$+ [J_2(1,1,n,p,r) - J_3(1,1,n,p,r,1)]\ U_5(m,q,s)$$

$$- [I_1(1,1,p,r,n) - I_2(1,1,p,r,n,1)]\ U_2(m,q,s)$$

$$+ [J_2(1,1,p,n,r) - J_3(1,1,p,n,r,1)]\ U_4(m,q,s)$$

$$+ [J_2(1,1,r,n,p) - J_3(1,1,r,n,p,1)]\ U_4(m,s,q)$$

$$- [I_3(1,1,n,p,r) - I_4(1,1,n,p,r,1)]\ U_6(m,q,s)\}$$

$$I^{1111}_{3335nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= npr\pi^4\, I_4(n,p,r,1)\, I_0(1,m,q,s)$$

$$I^{1122}_{3335nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= nps\pi^4\, I_2(1,r,n,p)\, I_2(m,q,s,1)$$

$$I^{1111}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= np\pi^4\, \{[J_4(r,n,p,1,1) + rJ_4(1,n,p,r,1)]\, I_1(1,1,m,q,s)$$
$$- [I_5(n,p,r,1,1) - rI_3(1,r,n,p,1)]\, J_0(1,1,m,q,s)\}$$

$$I^{1221}_{3335nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= I^{1122}_{3335srqpmn}$$

$$I^{1212}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= nq\pi^4\, \{[J_2(1,p,r,n,1) + rJ_2(1,1,p,n,r)]\, I_3(1,m,q,s,1)$$
$$- [I_3(1,p,n,r,1) - rI_1(1,1,p,r,n)]\, J_2(1,m,s,q,1)\}$$

$$I^{1221}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= -\, I^{1212}_{3345qpmnsr}$$

$$I^{1111}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^4 \,[B_1(1,n,p,r)\ I_2(1,1,1,m,q,s) - B_2(1,n,r,p)\ J_1(1,1,1,m,s,q)$$

$$- B_2(1,n,p,r)\ J_1(1,1,1,m,q,s) + B_3(1,n,r,p)\ J_0(1,1,1,m,q,s)]$$

$$I^{1122}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= n\pi^4 \,[C_1(1,n,r,p)\ C_4(m,1,q,s) - C_3(1,n,r,p)\ C_2(m,1,q,s)$$

$$- C_2(1,n,r,p)\ C_2(m,1,q,s) + C_4(1,n,r,p)\ C_1(m,1,q,s)]$$

$$I^{1221}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= n\pi^4 \,[I_2(1,1,p,r,n,1)\ B_4(1,m,q,s) - J_3(1,1,p,n,r,1)\ B_5(1,m,q,s)$$

$$- J_3(1,1,r,n,p,1)\ B_5(1,m,s,q) + I_4(1,1,n,r,p,1)\ B_6(1,m,q,s)]$$

$$I^{2222}_{3335nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= I^{1111}_{3335mnqpsr}$$

$$I^{1122}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= np\pi^4 \,(J_2(1,1,r,n,p)\ [I_3(m,q,s,1,1) - sI_1(1,m,q,s,1)]$$

$$- I_3(1,1,n,p,r)\ [J_2(m,q,s,1,1) + sJ_2(1,m,q,s,1)])$$

$$I^{2211}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= - I^{1122}_{3345mnqpsr}$$

$$I^{2222}_{3345nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= - I^{1111}_{3345mnqpsr}$$

$$I^{2112}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= I^{1221}_{3445mnqpsr}$$

$$I^{2211}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= I^{1122}_{3445mnqpsr}$$

$$I^{2222}_{3445nmpqrs} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= I^{1111}_{3445mnqpsr}$$

$$I^{1111}_{4445nmpqrs} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$\begin{aligned}
= \pi^4 \,[&E_1(1,n,p,r)\, D_1(1,m,q,s) - E_2(1,r,n,p)\, D_2(1,s,m,q) \\
&- E_2(1,p,n,r)\, D_2(1,q,m,s) + E_3(1,p,r,n)\, D_3(1,q,s,m) \\
&- E_2(1,n,p,r)\, D_2(1,m,q,s) + E_3(1,n,r,p)\, D_3(1,m,s,q) \\
&+ E_3(1,n,p,r)\, D_3(1,m,q,s) - E_4(1,n,p,r)\, D_4(1,m,q,s)]
\end{aligned}$$

$$I^{1122}_{4445nmpqrs} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$\begin{aligned} = \pi^4 [&G_6(1,r,n,p)\ F_6(1,s,m,q) - G_3(1,r,n,p)\ F_3(1,s,m,q) \\ &- G_5(1,r,p,n)\ F_5(1,s,q,m) + G_2(1,r,p,n)\ F_2(1,s,q,m) \\ &- G_5(1,r,n,p)\ F_5(1,s,m,q) + G_2(1,r,n,p)\ F_2(1,s,m,q) \\ &+ G_4(1,r,n,p)\ F_4(1,s,m,q) - G_1(1,r,n,p)\ F_1(1,s,m,q)] \end{aligned}$$

$$I^{1221}_{4445nmpqrs} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,1}(x,y)\,dxdy$$

$$= - I^{1122}_{4445srqpmn}$$

$$I^{2222}_{4445nmpqrs} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{5,2}(x,y)\,dxdy$$

$$= - I^{1111}_{4445mnqpsr}$$

$$I^{1111}_{3333nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,1}(x,y;r,s) f_{3,1}(x,y;t,u)\,dxdy$$

$$= nprt\pi^4\ I_4(n,p,r,t)\ I_0(m,q,s,u)$$

$$I^{1111}_{3334nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,1}(x,y;r,s) f_{4,1}(x,y;t,u)\,dxdy$$

$$\begin{aligned} = npr\pi^4 \{&[J_4(t,n,p,r,1) + tJ_4(1,n,p,r,t)]\ I_1(1,m,q,s,u) \\ &- [I_5(n,p,r,t,1) - tI_3(1,t,n,p,r)]\ J_0(1,m,q,s,u)\} \end{aligned}$$

$$I^{1122}_{3333nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,2}(x,y;r,s) f_{3,2}(x,y;t,u)\,dxdy$$

$$= npsu\pi^4\ I_2(r,t,n,p)\ I_2(m,q,s,u)$$

$$I^{1122}_{3334nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{3,2}(x,y;r,s) f_{4,2}(x,y;t,u)\, dxdy$$

$$= nps\pi^4 \{J_2(1,r,t,n,p)\ [I_3(m,q,s,u,1) - uI_1(1,m,q,u,s)]$$

$$- I_3(1,r,n,p,t)\ [J_2(m,q,u,s,1) + uJ_2(1,m,q,s,u)]\}$$

$$I^{1111}_{3344nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,1}(x,y;t,u)\, dxdy$$

$$= np\pi^4\ [B_1(n,p,r,t)\ I_2(1,1,m,q,s,u) - B_2(n,p,t,r)\ J_1(1,1,m,q,u,s)$$

$$- B_2(n,p,r,t)\ J_1(1,1,m,q,s,u) + B_3(n,p,r,t)\ I_0(1,1,m,q,s,u)]$$

$$I^{1122}_{3344nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u)\, dxdy$$

$$= np\pi^4\ [I_2(1,1,r,t,n,p)\ B_4(m,q,s,u) - J_3(1,1,r,n,p,t)\ B_5(m,q,s,u)$$

$$- J_3(1,1,t,n,p,r)\ B_5(m,q,u,s) + I_4(1,1,n,p,r,t)\ B_6(m,q,s,u)]$$

$$I^{1221}_{3334nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s) f_{4,1}(x,y;tu)\, dxdy$$

$$= - I^{1122}_{3334srqpmnut}$$

$$I^{1212}_{3344nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,2}(x,y;t,u)\, dxdy$$

$$= nq\pi^4\ [C_1(p,n,t,r)\ C_4(m,q,s,u) - C_3(p,n,t,r)\ C_3(m,q,s,u)$$

$$- C_2(p,n,t,r)\ C_2(m,q,s,u) + C_4(p,n,t,r)\ C_1(m,q,s,u)]$$

$$I^{1111}_{3444nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,1}(x,y;t,u)\,dxdy$$

$$\begin{aligned} = n\pi^4 \, [&E_1(n,p,r,t)\, D_1(m,q,s,u) - E_2(n,t,p,r)\, D_2(m,u,q,s) \\ &- E_2(n,r,p,t)\, D_2(m,s,q,u) + E_3(n,u,t,p)\, D_3(m,s,u,q) \\ &- E_2(n,p,r,t)\, D_2(m,q,s,u) + E_3(n,p,t,r)\, D_3(m,q,u,s) \\ &+ E_3(n,p,r,t)\, D_3(m,q,s,u) - E_4(n,p,r,t)\, D_4(m,q,s,u)] \end{aligned}$$

$$I^{1122}_{3444nmpqrstu} = \int_0^1 \int_0^1 f_{3,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u)\,dxdy$$

$$\begin{aligned} = n\pi^4 \, [&F_1(n,p,r,t)\, G_1(m,q,s,u) - F_2(n,p,r,t)\, G_2(m,q,s,u) \\ &- F_2(n,p,t,r)\, G_2(m,q,u,s) + F_3(n,p,r,t)\, G_3(m,q,s,u) \\ &- F_4(n,p,r,t)\, G_4(m,q,s,u) + F_5(n,p,r,t)\, G_5(m,q,s,u) \\ &+ F_5(n,p,t,r)\, G_5(m,q,u,s) - F_6(n,p,r,t)\, G_6(m,q,s,u)] \end{aligned}$$

$$I^{2222}_{3333nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s) f_{3,2}(x,y;t,u)\,dxdy$$

$$= I^{1111}_{3333mnqpsrut}$$

$$I^{2222}_{3334nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{3,2}(x,y;r,s) f_{4,2}(x,y;t,u)\,dxdy$$

$$= - I^{1111}_{3334mnqpsrut}$$

$$I^{2211}_{3344nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,1}(x,y;t,u)\,dxdy$$

$$= I^{1122}_{3344mnqpsrut}$$

$$I^{2222}_{3344nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{3,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u)\,dxdy$$

$$= I^{1111}_{3344mnqpsrut}$$

$$I^{2112}_{3444nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,2}(x,y;t,u)\,dxdy$$

$$= - I^{1122}_{3444mnutqpsr}$$

$$I^{2222}_{3444nmpqrstu} = \int_0^1 \int_0^1 f_{3,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u)\,dxdy$$

$$= - I^{1111}_{3444mnqpsrut}$$

$$I^{1111}_{4444nmpqrstu} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,1}(x,y;r,s) f_{4,1}(x,y;t,u)\,dxdy$$

$$\begin{aligned} = \pi^4 \, [&K_0(n,p,r,t)\, L_0(m,q,s,u) - K_1(t,n,p,r)\, L_1(u,m,q,s) \\ &- K_1(r,n,p,t)\, L_1(s,m,q,u) + K_2(r,t,n,p)\, L_2(s,u,m,q) \\ &- K_1(p,n,r,t)\, L_1(q,m,s,u) + K_2(p,t,n,r)\, L_2(q,u,m,s) \\ &+ K_2(p,r,n,t)\, L_2(q,s,m,u) - K_3(p,r,t,n)\, L_3(q,s,u,m) \\ &- K_1(n,p,r,t)\, L_1(m,q,s,u) + K_2(n,t,p,r)\, L_2(m,u,q,s) \\ &+ K_2(n,r,p,t)\, L_2(m,s,q,u) - K_3(n,r,t,p)\, L_3(m,s,u,q) \\ &+ K_2(n,p,r,t)\, L_2(m,q,s,u) - K_3(n,p,t,r)\, L_3(m,q,u,s) \\ &- K_3(n,p,r,t)\, L_3(m,q,s,u) + K_4(n,p,r,t)\, L_4(m,q,s,u)] \end{aligned}$$

$$I^{1122}_{4444nmpqrstu} = \int_0^1 \int_0^1 f_{4,1}(x,y;n,m) f_{4,1}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u) dxdy$$

$$= \pi^4 [M_{02}(n,p,r,t)\ M_{02}(s,u,m,q) - M_{01}(n,p,r,t)\ M_{10}(s,u,m,q)$$

$$- M_{01}(n,p,t,r)\ M_{10}(u,s,m,q) + M_{00}(n,p,r,t)\ M_{00}(s,u,m,q)$$

$$- M_{12}(p,n,r,t)\ M_{21}(s,u,q,m) + M_{11}(p,n,r,t)\ M_{11}(s,u,q,m)$$

$$+ M_{11}(p,n,t,r)\ M_{11}(u,s,q,m) - M_{10}(p,n,r,t)\ M_{01}(s,u,q,m)$$

$$- M_{12}(n,p,r,t)\ M_{21}(s,u,m,q) + M_{11}(n,p,r,t)\ M_{11}(s,u,m,q)$$

$$+ M_{11}(n,p,t,r)\ M_{11}(u,s,m,q) - M_{10}(n,p,r,t)\ M_{01}(s,u,m,q)$$

$$+ M_{22}(n,p,r,t)\ M_{22}(s,u,m,q) - M_{21}(n,p,r,t)\ M_{12}(s,u,m,q)$$

$$- M_{21}(n,p,t,r)\ M_{12}(u,s,m,q) + M_{20}(n,p,r,t)\ M_{02}(s,u,m,q)]$$

$$I^{2222}_{4444nmpqrstu} = \int_0^1 \int_0^1 f_{4,2}(x,y;n,m) f_{4,2}(x,y;p,q) f_{4,2}(x,y;r,s) f_{4,2}(x,y;t,u) dxdy$$

$$= I^{1111}_{4444mnqpsrut}$$

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